

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

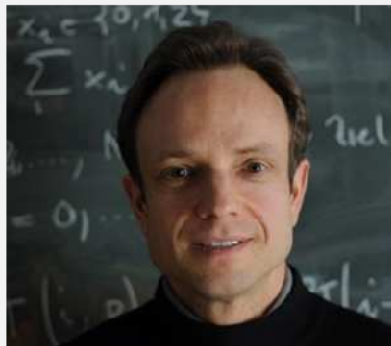
Course Video

Section 11.1: Initial-Value Problems (IVPs) for Ordinary Differential Equations (ODEs)

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Tablet notes may have been corrected and/or extended after the recording of the video!

Prerequisites.

- Concept of ODE from analysis course
- Basic results about existence and uniqueness of solutions of initial-value problems

Duration: minutes

XI. Numerical Integration - Single-Step Methods

11.1. Initial-Value Problems for ODEs

11.1.1. Ordinary Differential Equations (ODEs)

$$\frac{dy}{dt}(t) =: \dot{y}(t) = \overset{\substack{\text{"time variable"} \\ \downarrow}}{f(t, y(t))} \quad (\text{ODE})$$

$f: I \times D \rightarrow \mathbb{R}^N$ right-hand-side function
 $I \subset \mathbb{R}$ interval

$D \subset \mathbb{R}^N \triangleq$ *state space* (configuration space)

$y \in C^1(I, D)$ is a solution, if (ODE) satisfied $\forall t \in I$

For $N > 1$ $\dot{y} = f(t, y)$ can be viewed as a **system of ordinary differential equations**:

$$\dot{y} = f(t, y) \iff \begin{bmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_N \end{bmatrix} = \begin{bmatrix} f_1(t, y_1, \dots, y_N) \\ \vdots \\ f_N(t, y_1, \dots, y_N) \end{bmatrix}.$$

② Smoothness of solutions :

$$\left. \begin{array}{l} \gamma = \gamma(t) \text{ solution of } \dot{\gamma} = f(\gamma) \\ \text{on } I \subset \mathbb{R} \text{ and } f \in C^m(D, \mathbb{R}^N) \end{array} \right\} \Rightarrow \gamma \in C^{m+1}(I, D)$$

Linear ODEs

Scalar case ($N=1$) : $\dot{\gamma} = a\gamma$, $a \in \mathbb{R}$

$\Rightarrow$ One-parameter family of solutions: $\gamma(t) = ce^{at}$, $c \in \mathbb{R}$

$N > 1$: $\dot{\gamma} = A\gamma$, $A \in \mathbb{R}^{N,N}$

Diagonalization :

$$A = SDS^{-1}, \quad S \in \mathbb{R}^{N,N} \text{ regular}, \quad D = \text{diag}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{N,N}, \quad (11.1.1.13)$$

$$\dot{\mathbf{y}} = SDS^{-1}\mathbf{y} \Rightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z} \text{ with } \mathbf{z}(t) := S^{-1}\mathbf{y}(t).$$

Decoupled scalar ODEs: $\dot{z}_e(t) = \lambda_e z_e(t)$

$$\mathbf{y}(t) = S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix} S^{-1} \mathbf{w} \text{ for some } \mathbf{w} \in \mathbb{R}^N. \quad (11.1.1.14)$$

$\uparrow$
N-parameter family

11.1.2. Mathematical Modeling with ODEs

- Classical rigid-body mechanics : $\ddot{\gamma} = f(\gamma)$
- Electrical circuits, transients
- Chemical reaction kinetics

Ex. 11.1.2.5 (Population dynamics: predator-prey model)

$$N=2: \quad \gamma(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \rightarrow \begin{array}{l} \text{prey} \\ \text{predator} \end{array} \quad (\text{population densities})$$



State space $D = [0, \infty]^2 \subset \mathbb{R}^2$

ODE-based model: autonomous Lotka-Volterra ODE:

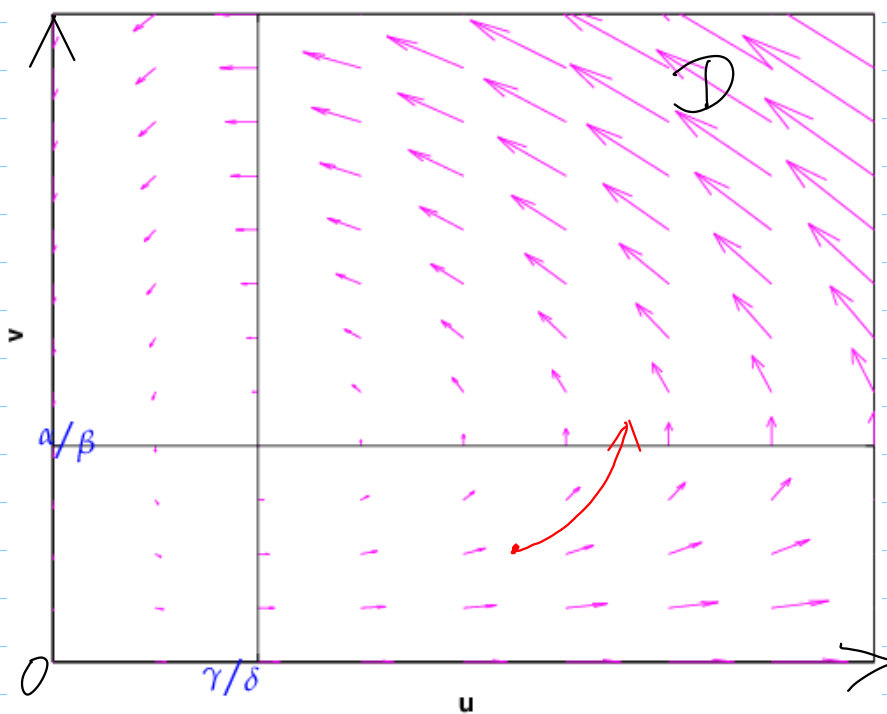
$$\begin{aligned} \dot{u} &= (\alpha - \beta v)u \\ \dot{v} &= (\delta u - \gamma)v \end{aligned} \Leftrightarrow \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) \text{ with } \mathbf{y} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} (\alpha - \beta v)u \\ (\delta u - \gamma)v \end{bmatrix}, \quad (11.1.2.6)$$

with positive model parameters $\alpha, \beta, \gamma, \delta > 0$.

$$v = 0 \Rightarrow \dot{u} = \alpha u \Rightarrow \text{exponential growth}$$

$$u = 0 \Rightarrow \dot{v} = -\gamma v \Rightarrow \text{exponential decay}$$

③



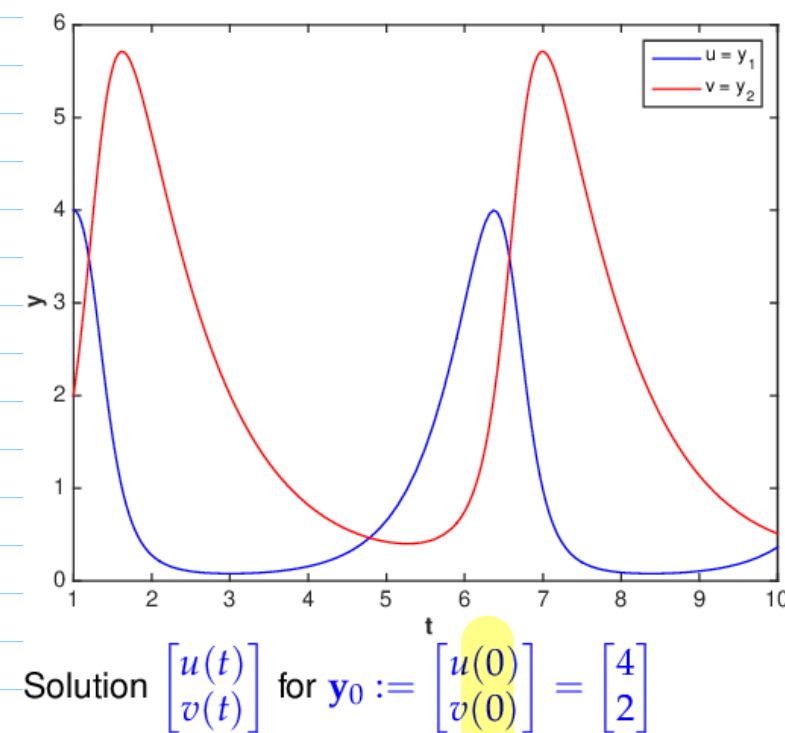
state space

$$\dot{\mathbf{y}} \triangleq \mathbf{f}(\mathbf{y})$$

a vector field
= velocity field

Need to impose initial values to obtain a unique solution

IVP

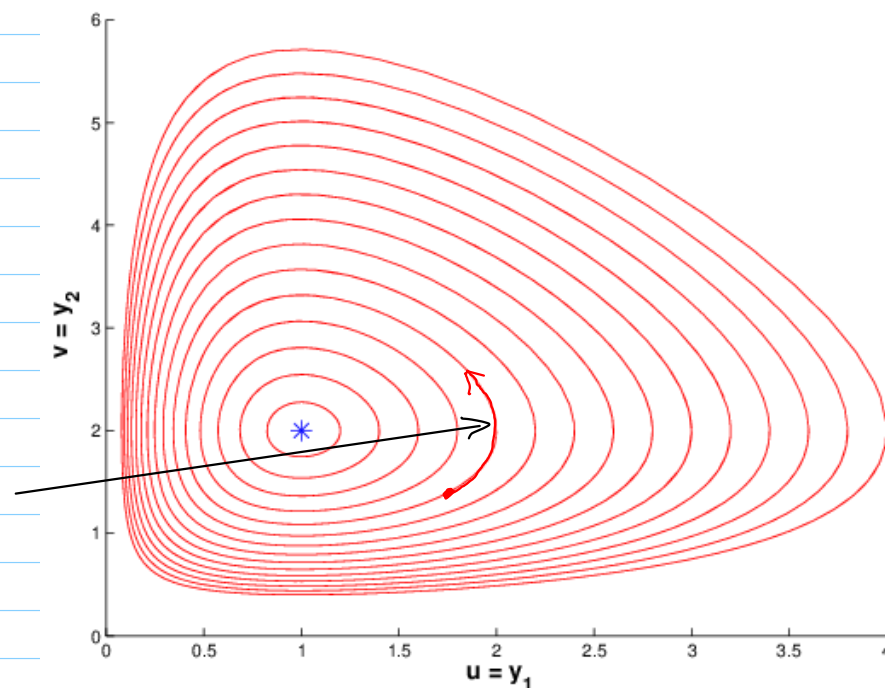


Why initial time $t_0 = 0$

Solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ for $\mathbf{y}_0 := \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$t \mapsto \mathbf{y}(t) \in \mathbb{R}^n$
 $\triangleq$ a curve in the plane

solution curve
(trajectory)



Solution curves for (11.1.2.6)

Definition 11.1.2.4. Autonomous ODE

An ODE of the form $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, that is, with a right hand side function that does not depend on time, but only on state, is called **autonomous**.

§ 11.1.3.3 (IVPs for autonomous ODEs)

$t \mapsto \mathbf{y}(t)$ solves $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$

$\triangleright t \mapsto \mathbf{y}(t-\tau)$ is also a solution $\forall \tau \in \mathbb{R}$

We can always fix initial time $t_0 = 0$!

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11.1.3. Theory of initial-value problems (IVPs)

A generic **Initial value problem** (IVP) for a **first-order ordinary differential equation** (ODE) $(\rightarrow,)$ can be stated as:

Find a function $y : I \rightarrow D$ that satisfies, cf. Def. 11.1.1.2,

$$\dot{y} = f(t, y), \quad y(t_0) = y_0. \quad (11.1.3.2)$$

- $f : I \times D \mapsto \mathbb{R}^N \triangleq$ **right hand side** (r.h.s.) ($N \in \mathbb{N}$),
- $I \subset \mathbb{R} \triangleq$ (time)interval $\leftrightarrow$ "time variable" t ,
- $D \subset \mathbb{R}^N \triangleq$ **state space/phase space** $\leftrightarrow$ "state variable" y ,
- $\Omega := I \times D \triangleq$ **extended state space** (of tuples (t, y)),
- $t_0 \in I \triangleq$ initial time, $y_0 \in D \triangleq$ initial state $\triangleright$ **initial conditions**.

mild smoothness requirement, satisfied $f \in C^\infty$

Theorem 11.1.3.17. Theorem of Peano & Picard-Lindelöf

If the right hand side function $f : \Omega \mapsto \mathbb{R}^N$ is locally Lipschitz continuous ($\rightarrow$ Def. 11.1.3.13) then for all initial conditions $(t_0, y_0) \in \Omega$ the IVP

$$\dot{y} = f(t, y), \quad y(t_0) = y_0. \quad (11.1.3.2)$$

has a solution $y \in C^1(J(t_0, y_0), \mathbb{R}^N)$ with **maximal** (temporal) domain of definition $J(t_0, y_0) \subset \mathbb{R}$.

existence & uniqueness

intrinsic life-span of solutions

Ex. 11.1.3.20 ("Explosion equation")

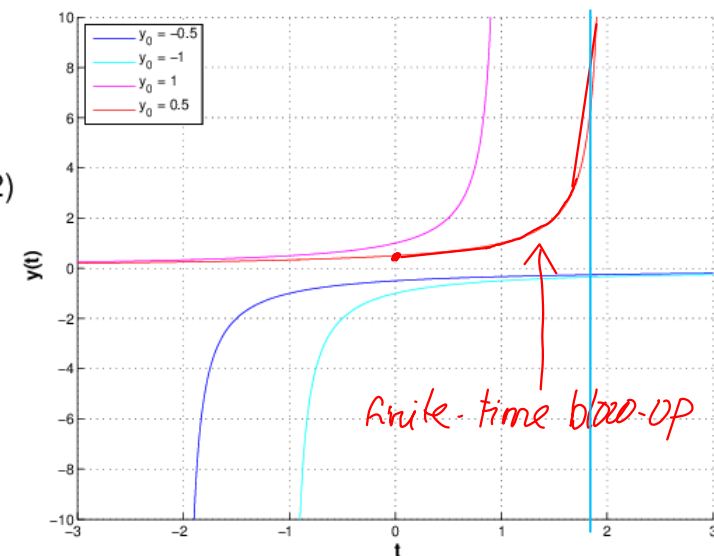
Scalar IVP: $\dot{y} = y^2, \quad y(0) = y_0 \in \mathbb{D} := \mathbb{R}$

We find the solutions

$$y(t) = \begin{cases} \frac{1}{y_0^{-1} - t} & , \text{ if } y_0 \neq 0, \\ 0 & , \text{ if } y_0 = 0, \end{cases} \quad (11.1.3.22)$$

with domains of definition

$$J(y_0) = \begin{cases}]-\infty, y_0^{-1}[& , \text{ if } y_0 > 0, \\ \mathbb{R} & , \text{ if } y_0 = 0, \\]y_0^{-1}, \infty[& , \text{ if } y_0 < 0. \end{cases}$$



§ 11.1.3.5 (Autonomization: Conversion into autonomous ODE)

Definition 11.1.2.4. Autonomous ODE

An ODE of the form $\dot{y} = f(y)$, that is, with a right hand side function that does not depend on time, but only on state, is called **autonomous**.

$$\dot{y} = f(t, y) : \quad z(t) := \begin{bmatrix} y(t) \\ t \end{bmatrix} \in \mathbb{D} \times \mathbb{I}$$

$$\dot{z}(t) = \begin{bmatrix} \dot{y}(t) \\ 1 \end{bmatrix} = \begin{bmatrix} f(z_{n+1}, z_1, \dots, z_n) \\ 1 \end{bmatrix} \triangleq \text{auton. ODE}$$

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Rem 11.1.3.6: From higher-order ODEs to first-order ODEs

2nd-order ODE (autonomous): $\ddot{y} = f(y)$

New unknown $z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \in \mathbb{R}^{2N}$

$$\dot{z}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} (z(t))_{N+1:2N} \\ f((z(t))_{1:N}) \end{bmatrix} =: g(z)$$

+ initial values $z(0) = z_0 \Leftrightarrow \begin{cases} y(0) = y_0 \in \mathbb{R}^N \\ \dot{y}(0) = v_0 \in \mathbb{R}^N \end{cases}$

For ODEs of order $n \in \mathbb{N}$ well-posed initial value problems need to specify initial values for the first $n - 1$ derivatives.

Main focus on 1st-order autonomous IVPs

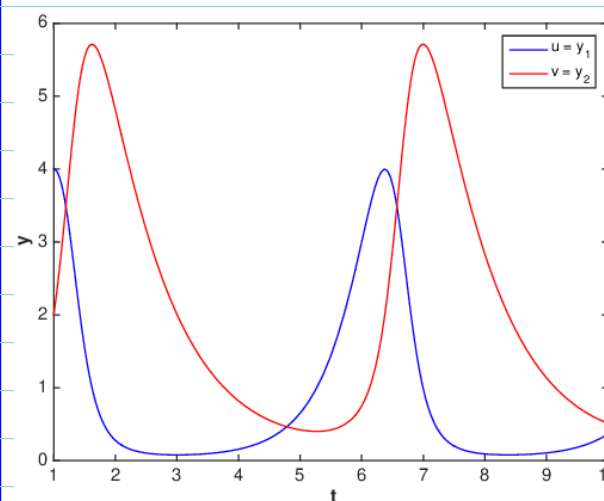
$$\dot{y} = f(y), \quad y(0) = y_0 \in \mathbb{R}^N$$

11.1.4. Evolution Operators

IVP: $\dot{y} = f(y), \quad y(0) = y_0 \in D \subset \mathbb{R}^N$ (11.1.3.18)
 $\uparrow \in C'$

Assumption 11.1.4.1. Global solutions

Solutions of (11.1.3.18) are global: $J(y_0) = \mathbb{R}$ for all $y_0 \in D$.

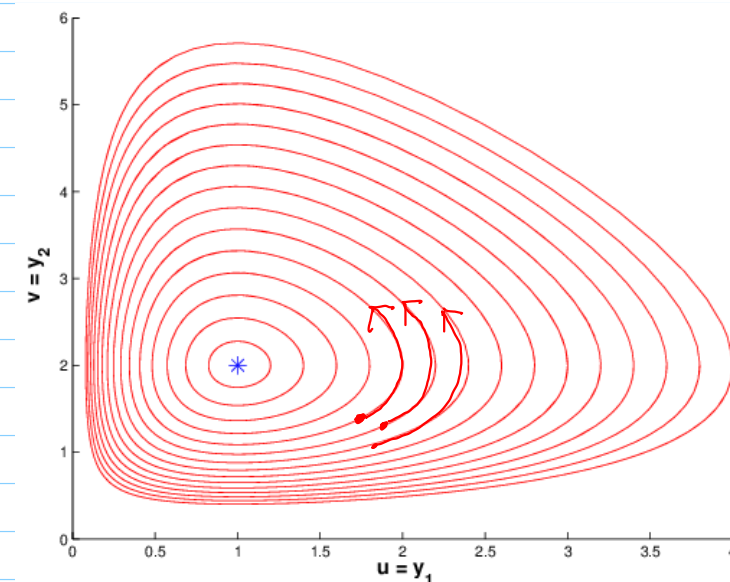


Solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ for $y_0 := \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Two perspectives

I.

$y_0 \Rightarrow t \in \mathbb{R} \mapsto y(t)$
(fixed)



trajectories $t \mapsto \Phi^t y_0$

II. Fix time t :

$y_0 \mapsto y(t)$
 $D \mapsto D$

Definition 11.1.4.3. Evolution operator/mapping

Under Ass. 11.1.4.1 the mapping

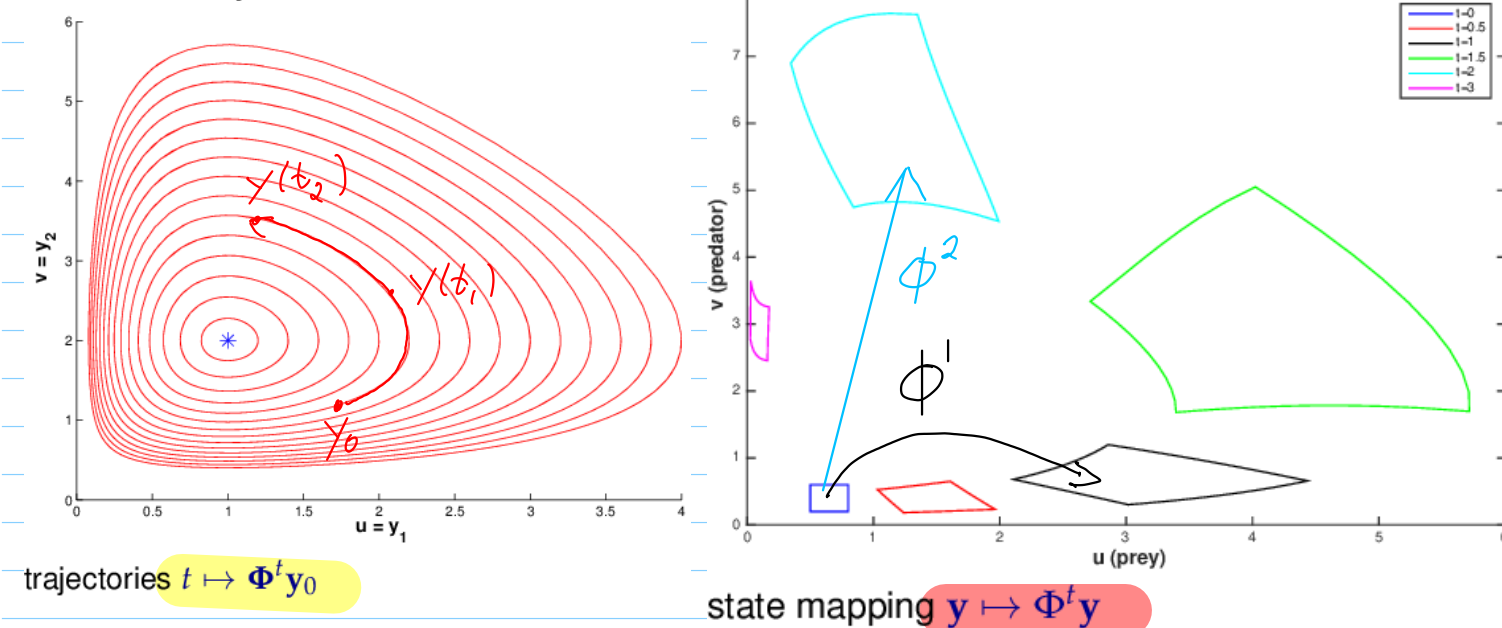
$$\Phi: \begin{cases} \mathbb{R} \times D & \mapsto D \\ (t, y_0) & \mapsto \Phi^t y_0 := y(t) \end{cases},$$

where $t \mapsto y(t) \in C^1(\mathbb{R}, \mathbb{R}^N)$ is the unique (global) solution of the IVP $\dot{y} = f(y)$, $y(0) = y_0$, is the **evolution operator**/mapping for the autonomous ODE $\dot{y} = f(y)$.

• Autonomous ODE

$$\Rightarrow \phi^s \circ \phi^t = \phi^{s+t} \quad (\text{group property})$$

Illustration for Ex. 11.1.2.5



$$\bullet \quad f(y) = \frac{\partial \Phi}{\partial t}(0, y) \Rightarrow \text{ODE from evol. op.}$$

▷ Please study *Review Questions 11.1.4.8* and try to answer them without relying on course material.

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