

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

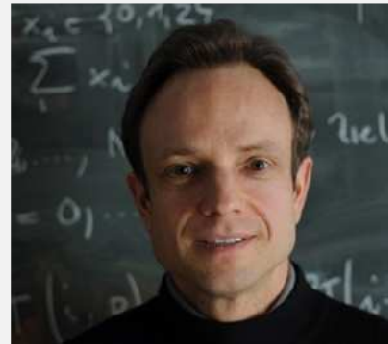
Course Video

Section 11.3.2: (Asymptotic) Convergence of Single-Step Methods

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Tablet notes may have been corrected and/or extended after the recording of the video!

Dependencies. Def. 6.2.2.7, Section 11.1.4, Section 11.3.1

Duration: minutes

XI Numerical Integration – Single-Step Methods

11.3 General Single-Step Methods

11.3.1 (Asymptotic) Convergence of SSMs

$$\text{IVP : } \dot{y} = f(y), \quad y(0) = y_0 \in \mathcal{D} \subset \mathbb{R}^N$$

$$\text{SSM: } y_{k+1} = \Psi^{h_k} y_k, \quad h_k := t_{k+1} - t_k$$

on mesh $\mathcal{M} = \{0 = t_0 < t_1 < \dots < t_m = T_0\}$

Focus : *h-convergence*, consider family of temporal meshes
with $h_m := \max_k h_k \longrightarrow 0$

$$\text{Discretization error : } \begin{aligned} & \cdot \| y_m - y(T) \| \\ & \cdot \max_k \| y_k - y(t_k) \| \end{aligned}$$

[$t \mapsto y(t) \triangleq \text{exact solution}$]

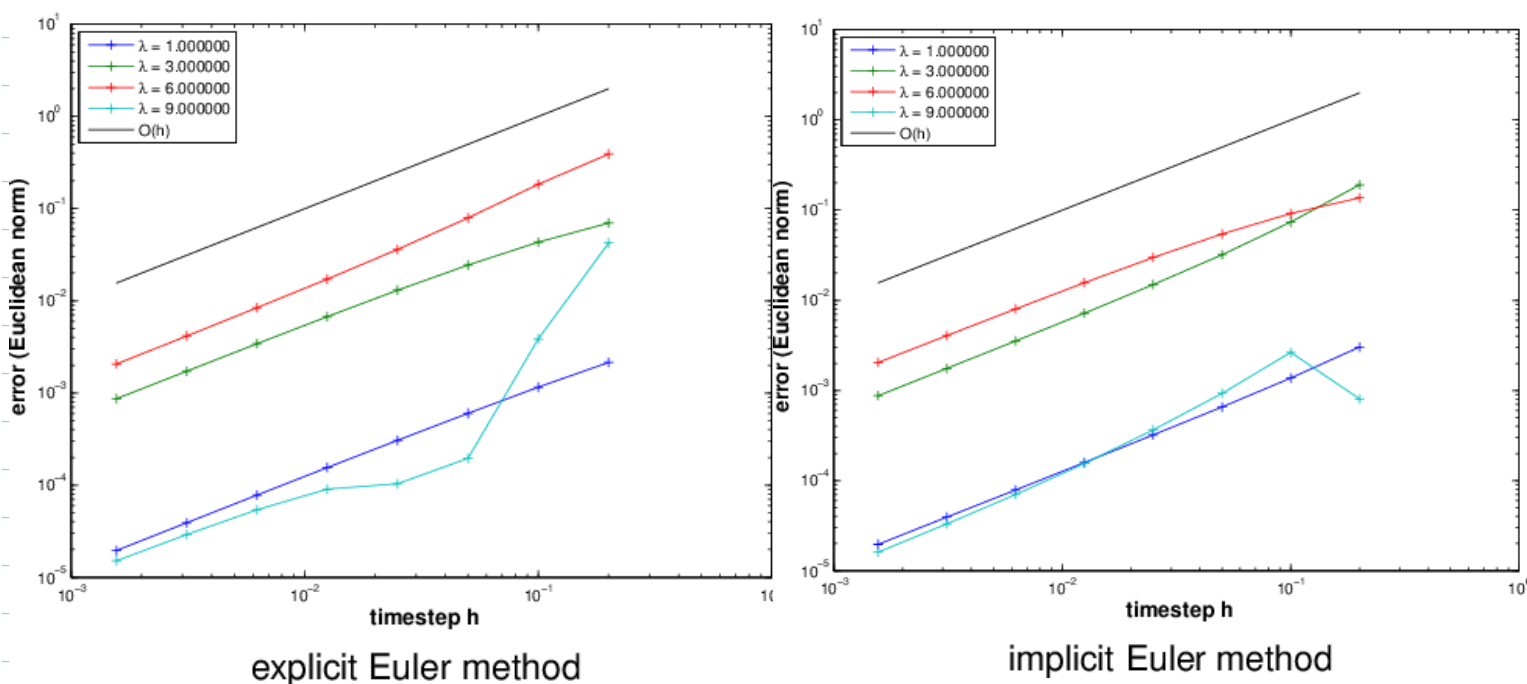
Exp. 11.3.2.5 : (Speed of convergence of polygonal methods)

◆ We consider the following IVP for the logistic ODE, see Ex. 6.1.2.1

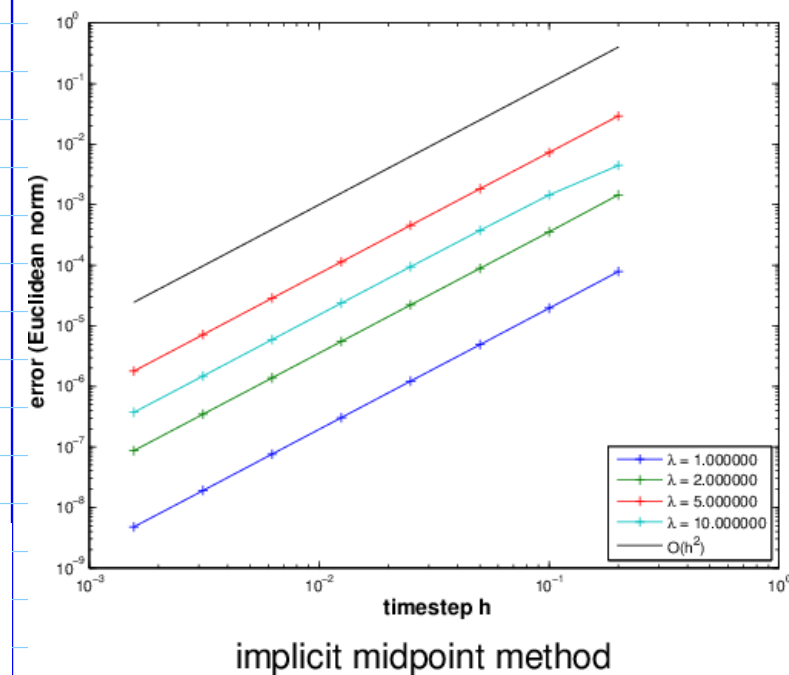
$$\dot{y} = \lambda y(1 - y) \quad , \quad y(0) = 0.01 \quad .$$

◆ We apply explicit and implicit Euler methods (6.2.1.4)/(6.2.2.2) with uniform timestep $h = 1/M$, $M \in \{5, 10, 20, 40, 80, 160, 320, 640\}$.

◆ Monitored: Error at final time $E(h) := |y(1) - y_M|$



▷ Empiric *algebraic convergence* : $E(h) = O(h)$
with rate/order 1 for $h \rightarrow 0$



▷ $E(h) = O(h^2)$
Alg. conv. w/ rate 2

Algebraic convergence of single step methods

Consider the numerical integration of an initial value problem

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \quad , \quad (11.1.3.2)$$

with sufficiently smooth right hand side function $\mathbf{f} : I \times D \rightarrow \mathbb{R}^N$.

Then conventional single step methods ($\rightarrow$ Def. 11.3.1.5) will enjoy *asymptotic algebraic convergence in the meshwidth*, more precisely, see ,

there is a $p \in \mathbb{N}$ such that the sequence $(\mathbf{y}_k)_k$ generated by the single step method for $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ on a mesh $\mathcal{M} := \{t_0 < t_1 < \dots < t_M = T\}$ satisfies

$$\max_k \|\mathbf{y}_k - \mathbf{y}(t_k)\| \leq Ch^p \quad \text{for} \quad h := \max_{k=1, \dots, M} |t_k - t_{k-1}| \rightarrow 0 \quad , \quad (11.3.2.7)$$

with $C > 0$ independent of $\mathcal{M}$

$p \neq$ rate of alg. conv.

↑ unknown!

Definition 11.3.2.8. Order of a single step method

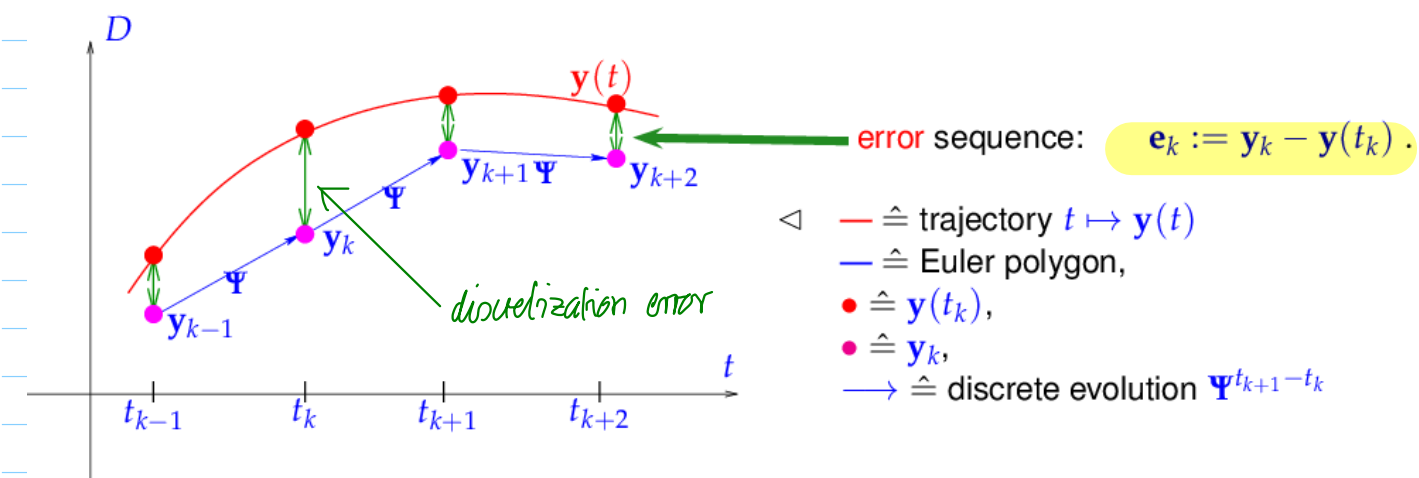
The maximal integer $p \in \mathbb{N}$ for which (11.3.2.7) holds for a single step method when applied to an ODE with (sufficiently) smooth right hand side, is called the **order** of the method.

§ 11.3.2.9 Convergence analysis for the explicit Euler method

- IVP : $\dot{y} = f(y)$, $y(0) = y_0 \in D$
- Global Lipschitz condition :

$$\exists L > 0: \|f(y) - f(z)\| \leq L\|y - z\| \quad \forall y, z \in D, \quad (11.3.2.10)$$

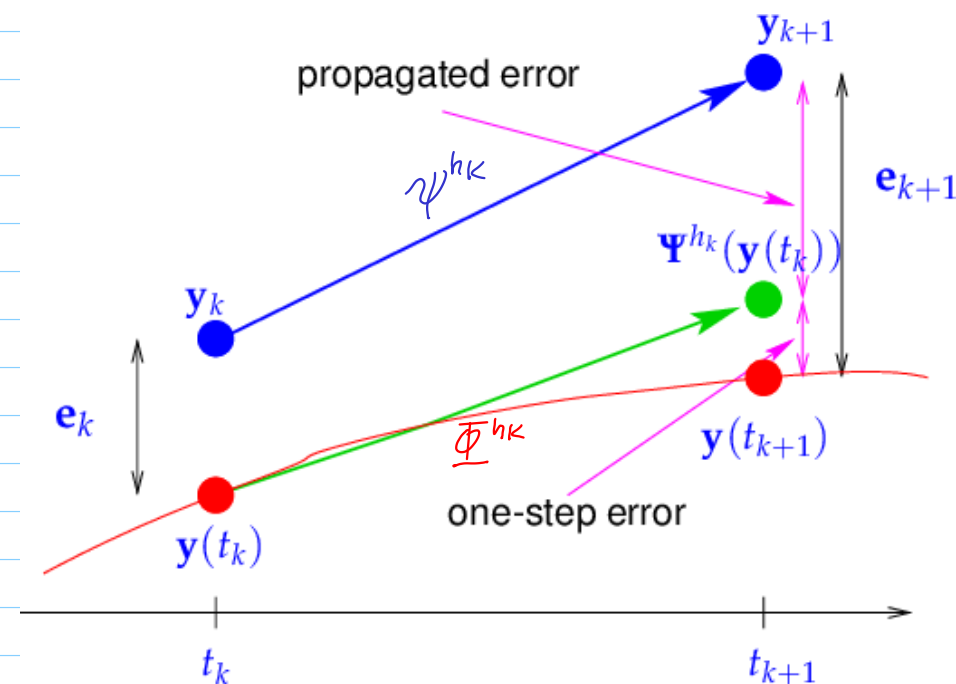
$$y_{k+1} = y_k + h_k f(y_k), \quad k = 1, \dots, M-1. \quad (11.2.1.5)$$



Fundamental error splitting:

$$\begin{aligned} e_{k+1} &= \Psi^{h_k} y_k - \Phi^{h_k} y(t_k) = y_{k+1} - y(t_{k+1}) \\ &= \underbrace{\Psi^{h_k} y_k - \Psi^{h_k} y(t_k)}_{\text{propagated error}} + \underbrace{\Psi^{h_k} y(t_k) - \Phi^{h_k} y(t_k)}_{\text{one-step error}}. \end{aligned} \quad (11.3.2.12)$$

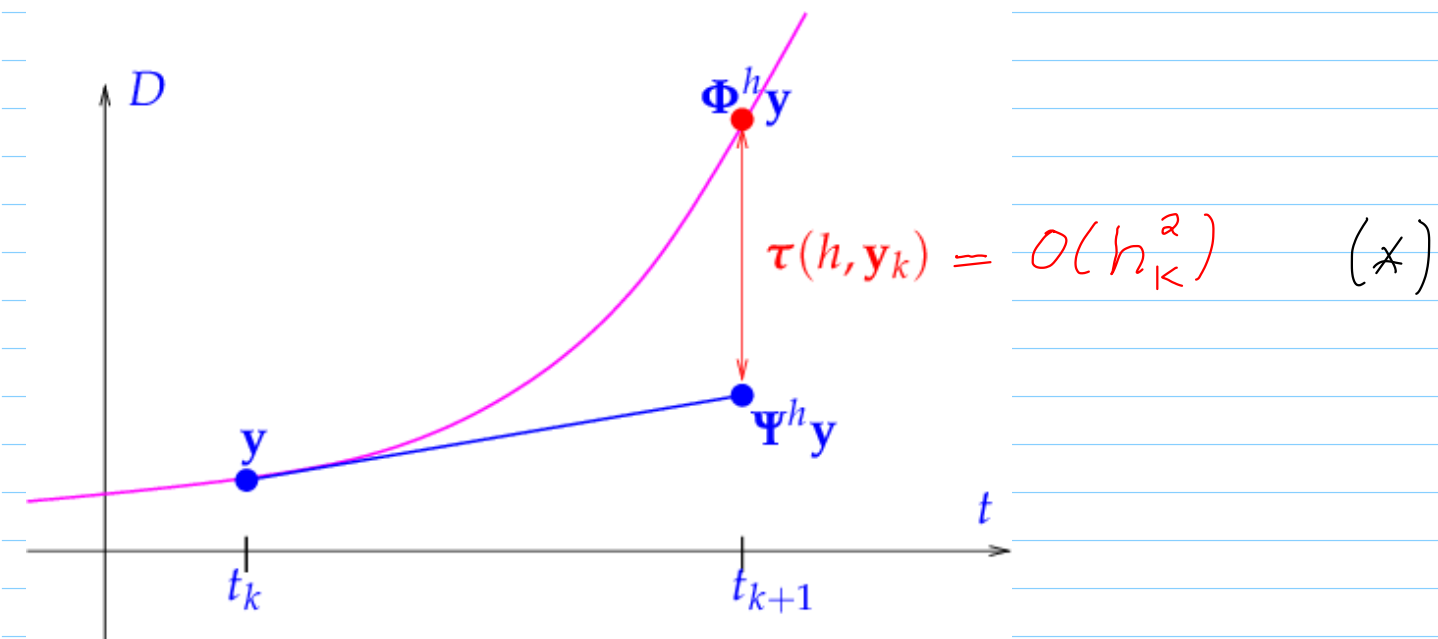
$\Psi \hat{=}$ discrete evol. op.
 $\Phi \hat{=}$ (exact) evol. op.



goal: recursion for e_k

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- Estimate for one-step error:



- Estimate for propagated error: ($h := h_k$)

$$\begin{aligned}
 \|\Psi^h \mathbf{y}_k - \Psi^h \mathbf{y}(t_k)\| &\stackrel{\text{expl. Euler}}{=} \|\mathbf{y}_k + h f(\mathbf{y}_k) - \mathbf{y}(t_k) - h f(\mathbf{y}(t_k))\| \\
 &\leq \|\mathbf{y}_k - \mathbf{y}(t_k)\| + h \|f(\mathbf{y}_k) - f(\mathbf{y}(t_k))\| \\
 &\stackrel{\text{L.C.}}{\leq} (1 + hL) \underbrace{\|\mathbf{y}_k - \mathbf{y}(t_k)\|}_{= \|e_k\|}
 \end{aligned}$$

$$\|e_{k+1}\| \leq (1 + h_k L) \|e_k\| + \tau(h_k, \mathbf{y}_k)$$

$$\text{Using } (*) = \dots \Rightarrow \|e_k\| = O(h_m)$$

Total error arises from accumulation of propagated one-step errors!

Remains valid for all consistent SSMs:

Order of algebraic convergence of single-step methods

Consider an IVP (11.1.3.2) with solution $t \mapsto \mathbf{y}(t)$ and a single step method defined by the discrete evolution Ψ ($\rightarrow$ Def. 11.3.1.5). If the *one-step error along the solution trajectory* satisfies (Φ is the evolution map associated with the ODE, see Def. 11.1.4.3)

$$\|\Psi^h \mathbf{y}(t) - \Phi^h \mathbf{y}(t)\| \leq Ch^{p+1} \quad \forall h \text{ sufficiently small, } t \in [0, T], \quad (11.3.2.22)$$

for some $p \in \mathbb{N}$ and $C > 0$, then, usually,

$$\max_k \|\mathbf{y}_k - \mathbf{y}(t_k)\| \leq \bar{C} h_{\mathcal{M}}^p,$$

with $\bar{C} > 0$ independent of the temporal mesh $\mathcal{M}$: The (pointwise) discretization error *converges algebraically* with *order/rate* p .

one-step error

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▷ Please tackle review questions 11.3.2.34
without looking at notes

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