

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

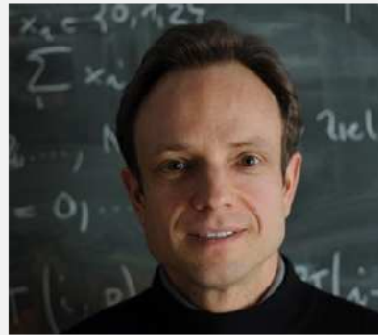
Course Video

Section 11.3: General Single-Step Methods

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Tablet notes may have been corrected and/or extended after the recording of the video!

Dependencies. Section 11.1.4, Section 11.2

Duration: minutes

XI Numerical Integration - Single-Step Methods

11.3 General Single-Step Methods

11.3.1 Definition

Wlog: restriction to autonomous ODEs $\dot{y} = f(y)$

11.2.1: Given temporal mesh $M = \{0 = t_0 < t_1 < \dots < t_m\}$,
explicit Euler produces sequence of states

$$y_0, y_1, y_2, \dots, y_m$$

by $y_{k+1} = y_k + h_k f(y_k), h_k := t_{k+1} - t_k$

More general: $y_{k+1} = \Psi(h_k, y_k) =: \Psi^{h_k} y_k$

with mapping $\Psi: \mathbb{R} \times D \rightarrow D$



Expl. Euler: $\Psi^T y = y + h f(y)$

② § 11.3.1.1. Discrete evolution operators

Expl. Euler : $y_{k+1} = \Psi^{h_k} y_k$
 $y(t_{k+1}) = \Phi^{h_k} y(t_k)$
 $t \mapsto y(t)$ solution of IVP w/ $y(0) = y_0$
 evolution operator $\Phi : I \times D \rightarrow D, I = [-\delta, \delta]$

$\triangleright \Psi \triangleq$ discrete evolution operator
 supposed to approximate $\Phi : \Psi \approx \Phi$

Numerical integration is concerned with the approximation of evolution operators.

Definition 11.3.1.5. Single step method (for autonomous ODE) $\rightarrow$

Given a discrete evolution $\Psi : \Omega \subset \mathbb{R} \times D \mapsto \mathbb{R}^N$, an initial state y_0 , and a temporal mesh $\mathcal{M} := \{0 =: t_0 < t_1 < \dots < t_M := T\}$, $M \in \mathbb{N}$, the recursion

$$y_{k+1} := \Psi(t_{k+1} - t_k, y_k), \quad k = 0, \dots, M-1, \quad (11.3.1.6)$$

defines a **single-step method** (SSM) for the autonomous IVP $\dot{y} = f(y)$, $y(0) = y_0$ on the interval $[0, T]$.

Rem: Ψ may be "implicit"

e.g. impl. Euler : $\Psi^h y : \Psi^h y = y + h f(\Psi^h y)$

§ 11.3.1.8. Consistent SSMs

$\Psi \approx \Phi$? Recall $\frac{d}{dt} \Phi^t y|_{t=0} = f(y)$, $y \in D$
 $\Rightarrow$ Want $\frac{d}{dh} \Psi^h y|_{h=0} = f(y)$

Minimal requirement: **Consistency**

Consistent discrete evolution

The discrete evolution Ψ defining a single step method according to Def. 11.3.1.5 and (11.3.1.6) for the autonomous ODE $\dot{y} = f(y)$ must be of the form

$$\Psi^h y = y + h \psi(h, y) \quad \text{with} \quad \begin{aligned} &\psi : I \times D \rightarrow \mathbb{R}^N \text{ continuous,} \\ &\psi(0, y) = f(y). \end{aligned} \quad (11.3.1.11)$$

$$\begin{aligned} \frac{d}{dh} \Psi^h y|_{h=0} &= \psi(h, y) + h \frac{\partial \psi}{\partial h}(h, y)|_{h=0} \\ &= \psi(0, y) = f(y) \end{aligned}$$



Expl. Euler : $\Psi(h, y) = y + h f(y)$
 $\Psi^T y = y + h f(y)$

③ Ex. 11.3.1.13 : Consistency of implicit midpoint method

Assume : $\underline{f}$ smooth

$$\begin{aligned} \Psi^h \underline{y} &= \underline{y} + h \underline{f} \left(\frac{1}{2} (\underline{y} + \Psi^h \underline{y}) \right) \\ \Rightarrow &= \underline{y} + h \underbrace{\underline{f} \left(\underline{y} + \frac{1}{2} h \underline{f} \left(\frac{1}{2} (\underline{y} + \Psi^h \underline{y}) \right) \right)}_{=: \psi(h, \underline{y})} \\ &=: \psi(h, \underline{y}) \end{aligned}$$

- $\psi(0, \underline{y}) = \underline{f}(\underline{y})$
- Implicit function thm. $\Rightarrow h \mapsto \Psi^h \underline{y}$ continuous

§ 11.3.1.16 : Output of SSMs

$\hat{=}$ sequence of states $(\underline{y}_k)_{k=0}^M$, but no trajectory?
interpolate to obtain function $[0, T] \rightarrow \mathbb{D}$

$$\dot{\underline{y}}(t_k) = \underline{f}(\underline{y}(t_k)) \approx \underline{f}(\underline{y}_k)$$

$\rightarrow$ may also be based on $\dot{\underline{y}}(t_k)$

e.g.: p.w. cubic Hermite interpolation

$\triangleright$ Please try to answer the review questions
 11.3.1.17 without aids.

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