

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

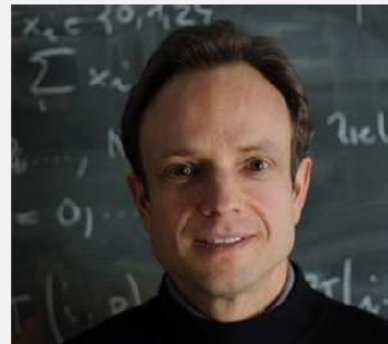
Course Video

Section 12.2: Stiff Initial-Value Problems

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Tablet notes may have been corrected and/or extended after the recording of the video!

Dependencies. Section 11.1, Section 12.1

Duration: minutes

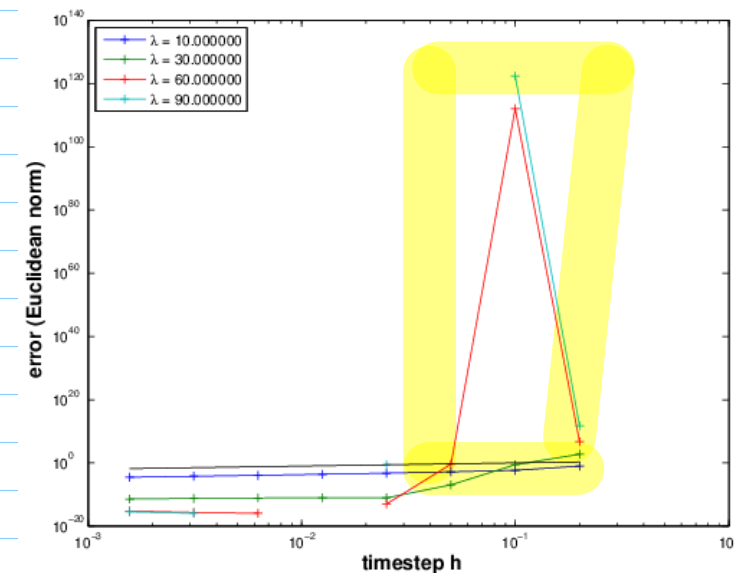
XII. Single-Step Methods for Stiff Initial-Value Problems

12.2 Stiff Initial-Value Problems

Sect. 12.1: Linear model problem analysis for
explicit RK-SSMs

→ Quantification of stability-induced timestep
constraints for (linear) IVPs with decaying solutions.

Ex. 12.1.0.3 (Blow-up of explicit Euler method)



Logistic ODE

$$\dot{y} = \lambda y(1-y)$$

$$y(0) = 0.01$$

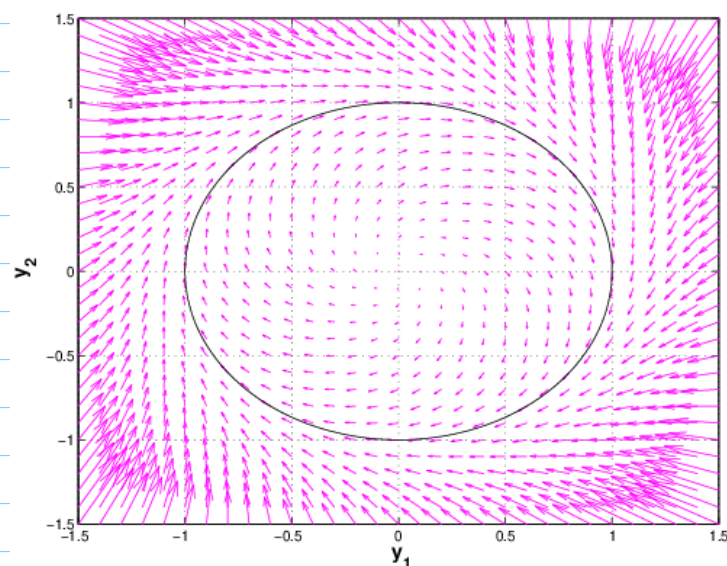
△ Convergence of expl.
Euler

λ large: blow-up of y_k for large timestep h

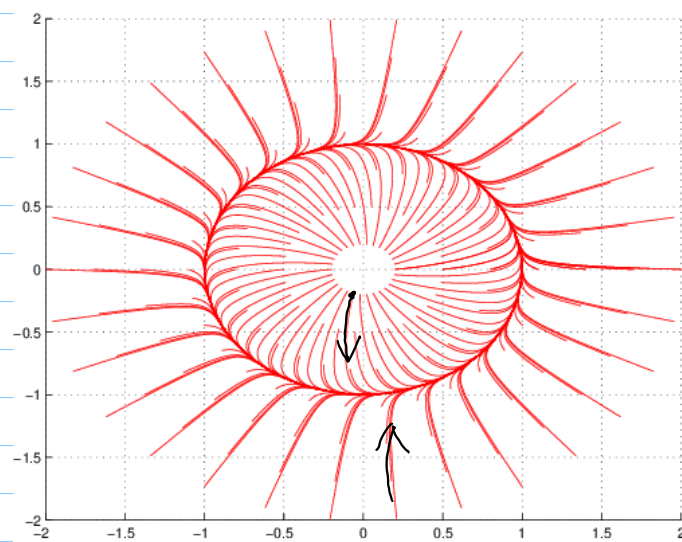
② Ex. 12.2.0.4 (Strongly attractive limit cycle)

$$\dot{y} = f(y) := \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y}_{\text{"rotation ODE"}} + \underbrace{\lambda(1 - \|y\|^2) y}_{\text{non-linear, } = 0 \text{ for } \|y\| = 1}, \quad D = \mathbb{R}^2$$

$$\|y(0)\|_2 = 1 \quad \triangleright \quad y(t) = \begin{bmatrix} \cos(t - \varphi) \\ \sin(t - \varphi) \end{bmatrix}, \quad t \in \mathbb{R}$$



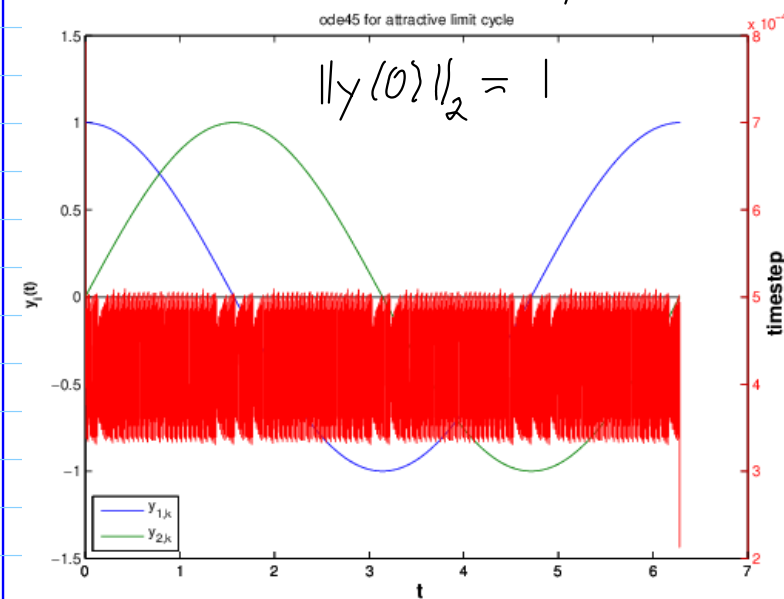
vectorfield f ($\lambda = 1$)



solution trajectories ($\lambda = 10$)

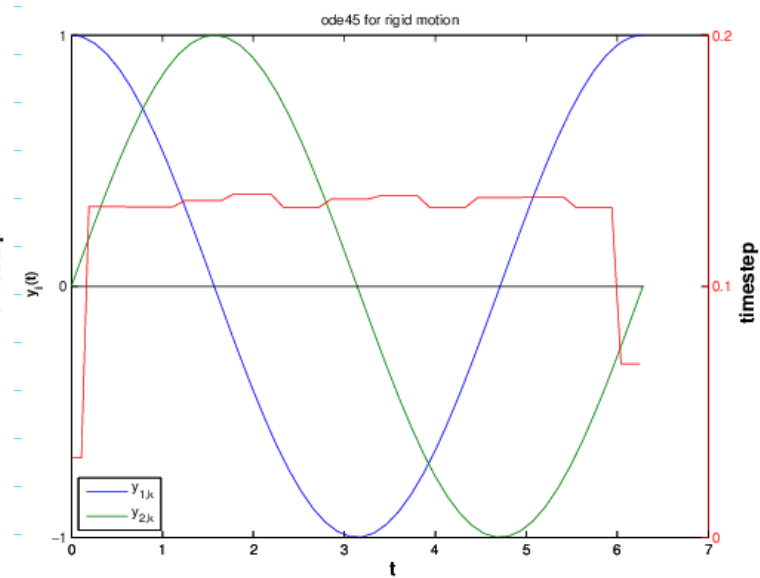
$\lambda \gg 1 \Rightarrow y(t)$ fast approach the unit circle

Ode45 adaptive explicit RK-SSM:



many (3794) steps ($\lambda = 1000$)

stability-induced timestep constraint



accurate solution with few steps ($\lambda = 0$)

Notion 12.2.0.7. Stiff IVP

An initial value problem is called **stiff**, if stability imposes much tighter timestep constraints on *explicit single step methods* than the accuracy requirements.

③ §12.2.0.8 Linearization of ODEs

Autonomous ODE : $\dot{y} = f(y)$, $f: D \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$ smooth
 $y^* \in D$: $t \rightarrow y(t)$ solution w/ $y(0) = y^*$

Linearization around y^* :

$$z(t) = y(t) - y^* : z(0) = 0$$

$$\dot{z} = f(z + y^*) = f(y^*) + Df(y^*)z + O(\|z\|^2)$$

► The short-time evolution of y with $y(0) = y^*$ is approximately governed by the affine-linear ODE

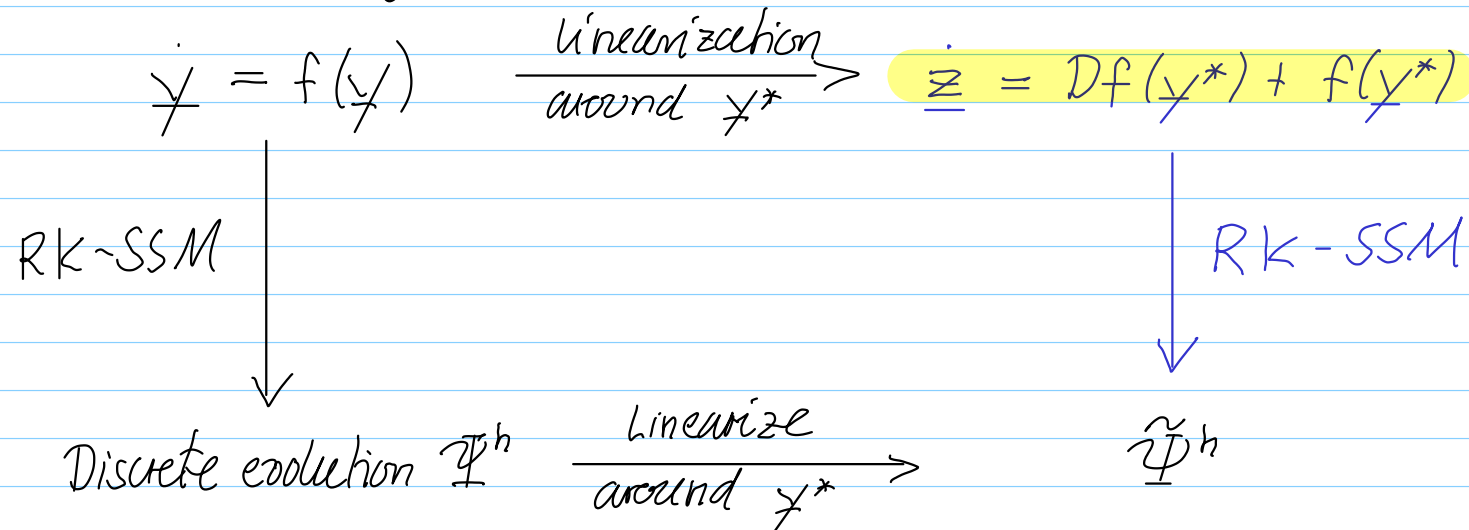
$$\dot{y} = M(y - y^*) + b, \quad M := Df(y^*) \in \mathbb{R}^{N,N}, \quad b := f(y^*) \in \mathbb{R}^N. \quad (12.2.0.10)$$

Ex 12.1.0.3 cont'd : $f(y) = \lambda y(1-y)$, $\lambda \in \mathbb{R}$
 $y^* = 1$ [$f(1) = 0$!]

$$\triangleright y(0) \approx y^* \Rightarrow \underbrace{\dot{z} \approx -\lambda z}_{\text{decay equation}}$$

§12.2.0.11 (Linearization of explicit RK-SSMs)

A commuting diagram: (timestep $h > 0$ fixed)



The discrete evolution of the RK-SSM for $\dot{y} = f(y)$ in the state y^* is close to the discrete evolution of the same RK-SSM applied to the linearization (12.2.0.10) of the ODE in y^* .

$$\left. \begin{array}{l} \dot{z} = Mz + b \\ z(0) = z_0 \end{array} \right\} \xrightarrow{\text{RK-SSM}} (z_k)$$

$$\Rightarrow \underline{w}_k = z_k - z_0 + M^{-1}b$$

$$\left. \begin{array}{l} \dot{w} = Mw \\ w(0) = M^{-1}b \end{array} \right\} \xrightarrow{\text{RK-SSM}} (w_k)$$

The behavior of an explicit Runge-Kutta single-step method applied to $\dot{y} = f(y)$ close to the state y^* is determined by the eigenvalues of the Jacobian $Df(y^*)$.

$t \mapsto \mathbf{y}(t) \stackrel{?}{=} \text{solution of IVP}$

How to distinguish stiff initial value problems

An initial value problem for an autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ will probably be stiff, if, for substantial periods of time,

$$\min\{\operatorname{Re} \lambda : \lambda \in \sigma(D\mathbf{f}(\mathbf{y}(t)))\} \ll 0, \quad (12.2.0.13)$$

and $\max\{\operatorname{Re} \lambda : \lambda \in \sigma(D\mathbf{f}(\mathbf{y}(t)))\} \lesssim 0, \quad (12.2.0.14)$

where $t \mapsto \mathbf{y}(t)$ is the solution trajectory and $\sigma(\mathbf{M})$ is the spectrum of the matrix $\mathbf{M}$, see Def. 9.1.0.1.

(13) $\Rightarrow \exists$ negative EV λ of large modulus

(14) $\Rightarrow$ no EV with large positive real part

$h\lambda \in S_{\psi} \Rightarrow$ stability-induced
time step constraint
for expl. RK-SSM

Ex. 12.2.0.15 (Predicting stiffness of non-linear IVPs)

Ex. 12.2.0.1: $N = 1$

IVP considered: $\dot{y} = f(y) := \lambda y^2(1 - y)$, $\lambda := 500$, $y(0) = \frac{1}{100}$.

We find ($y^* = 1$)

$$f'(y) = \lambda(2y - 3y^2) \Rightarrow f'(1) = -\lambda.$$

Ex. 12.2.0.4: $N = 2$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \lambda(1 - \|\mathbf{y}\|^2) \mathbf{y}, \quad \|\mathbf{y}_0\|_2 = 1. \quad (12.2.0.5)$$

satisfies $\|\mathbf{y}(t)\|_2 = 1$ for all times.

$$D\mathbf{f}(\mathbf{y}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \lambda(-2\mathbf{y}\mathbf{y}^T + (1 - \|\mathbf{y}\|_2^2)\mathbf{I}).$$

$$\sigma(D\mathbf{f}(\mathbf{y})) = \left\{ \underbrace{-\lambda - \sqrt{\lambda^2 - 1}}_{\ll 0}, \underbrace{-\lambda + \sqrt{\lambda^2 - 1}}_{\approx 0} \right\}, \text{ if } \|\mathbf{y}\|_2 = 1.$$

$$\lambda \gg 1 \Rightarrow \ll 0 \quad \approx 0$$

$\hookrightarrow$ timestep constraint

Rem. 12.2.0.16 (Characteristics of phenomena leading to stiff model IVPs)

$$\mathbf{f}(\mathbf{y}^*) = 0 \Leftrightarrow \text{fixed point}$$

Typical features of stiff IVPs:

- ◆ Presence of **fast transients** in the solution, see Ex. 12.1.0.3, Ex. 12.1.0.35,
- ◆ Occurrence of **strongly attractive** fixed points/limit cycles, see Ex. 12.2.0.4

⑤

▷ Please study the *Review Questions 12.2.0.17*
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