

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

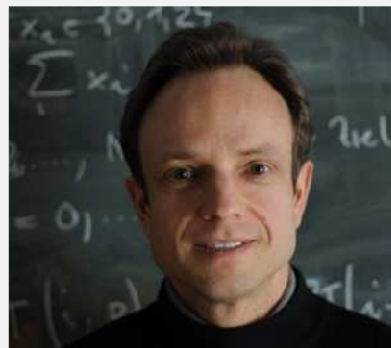
Course Video

Section 12.5: Splitting Methods

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Date: November 15, 2022

(C) Seminar für Angewandte Mathematik, ETH Zürich



Tablet notes may have been corrected and/or extended after the recording of the video!

Prerequisites.

Dependencies. Section 11.3, Section 11.3.2

Duration: minutes

XII. Single-Step Methods for Stiff Initial-Value Problems

12.5. Splitting Methods

§ 12.5.0.1 (Splitting idea: Composition of partial evolutions)

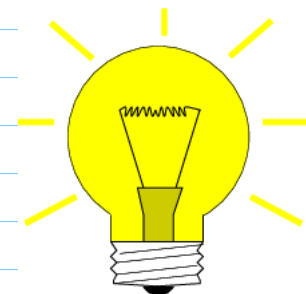
$$\text{ODE} : \dot{y} = f(y) + g(y) \quad , \quad f, g : D \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$$

Evolution operators:

$$\Phi_f : I \times D \longrightarrow D \iff \dot{y} = f(y)$$

$$\Phi_g : I \times D \longrightarrow D \iff \dot{y} = g(y)$$

↳ Assumed to be available



Idea: Build single step methods ($\rightarrow$ Def. 11.3.1.5) based on the following discrete evolutions

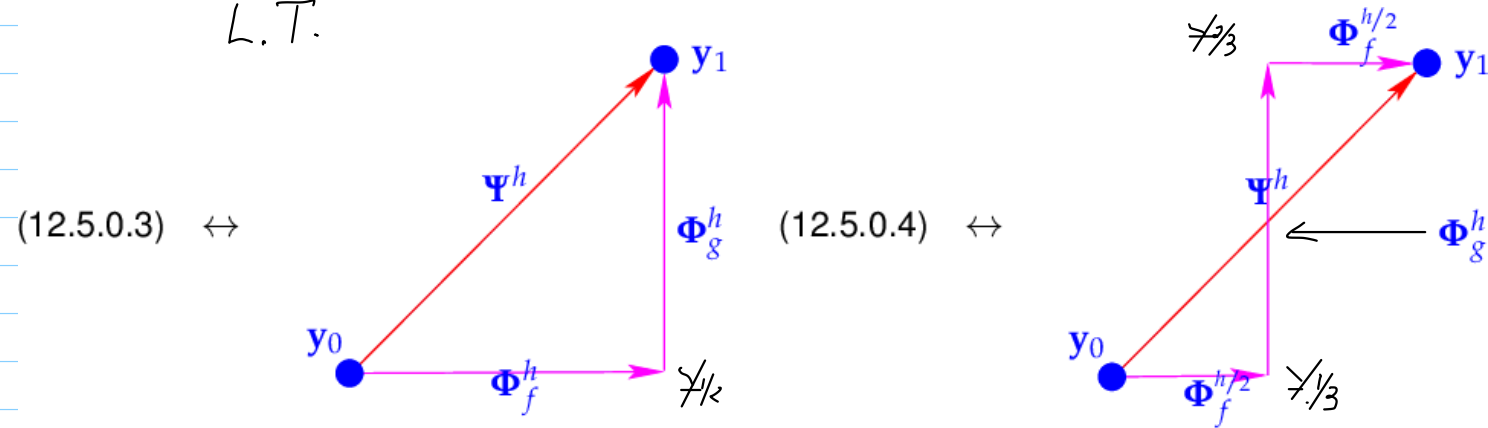
Lie-Trotter splitting: $\Psi^h = \Phi_g^h \circ \Phi_f^h, \quad (12.5.0.3)$

Strang splitting: $\Psi^h = \Phi_f^{h/2} \circ \Phi_g^h \circ \Phi_f^{h/2}. \quad (12.5.0.4)$

② ▷ Lie-Trotter splitting SSM, timestep $h > 0$

$$\begin{aligned} y_0 &\longrightarrow y_{1/2} := \Phi_f^h y_0 \longrightarrow y_1 := \Phi_g^h y_{1/2} \\ y_1 &\longrightarrow y_{3/2} := \Phi_f^h y_1 \longrightarrow y_2 := \Phi_g^h y_{3/2} \end{aligned}$$

L.T.



▷ Strang splitting SSM, timestep $h > 0$

$$\begin{aligned} y_0 &\longrightarrow y_{1/3} := \Phi_f^{h/2} y_0 \longrightarrow y_{2/3} := \Phi_g^h y_{1/3} \longrightarrow y_1 := \Phi_f^{h/2} y_{2/3} \\ y_1 &\longrightarrow y_{4/3} := \Phi_f^{h/2} y_1 \longrightarrow y_{5/3} := \Phi_g^h y_{4/3} \longrightarrow y_2 := \Phi_f^{h/2} y_{5/3} \\ y_2 &\longrightarrow y_{7/3} := \Phi_f^{h/2} y_2 \longrightarrow \dots \end{aligned}$$

$$\Phi_f^\tau \circ \Phi_f^b = \Phi_f^{\tau+b} \quad \triangleright \quad y_{4/3} := \Phi_f^h y_{2/3}, \quad y_{7/3} := \Phi_f^h y_{5/3}$$

Implementation:

$$\begin{aligned} y_0 &\xrightarrow{\Phi_f^{h/2}} y_{1/3} \xrightarrow{\Phi_g^h} y_{2/3} \xrightarrow{\Phi_f^{h/2}} y_{5/3} \xrightarrow{\Phi_g^h} y_{7/3} \dots \\ \dots &\xrightarrow{\Phi_f^{h/2}} y_0 \xrightarrow{\Phi_g^h} y_0 \xrightarrow{\Phi_f^{h/2}} y_m \end{aligned}$$

▷ L.T. & Strang differ only in the first & last step

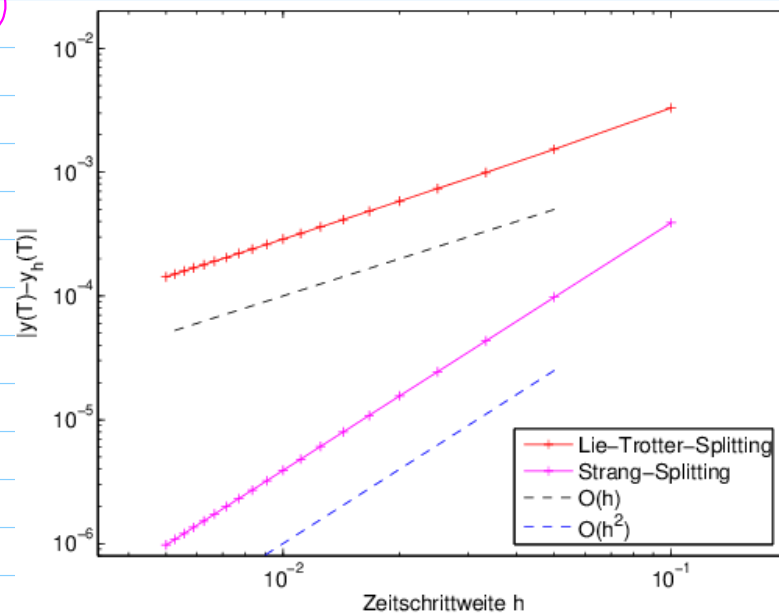
Exp. 12.5.0.5 (Convergence of simple splitting methods)

$$\text{IVP: } \dot{y} = \underbrace{\lambda y(1-y)}_{=:f(y)} + \underbrace{\sqrt{1-y^2}}_{=:g(y)}, \quad y(0) = 0.$$

$$\blacktriangleright \Phi_f^t y = \frac{1}{1 + (y^{-1} - 1)e^{-\lambda t}}, \quad t > 0, y \in]0, 1] \quad (\text{logistic ODE (11.1.2.2)})$$

$$\blacktriangleright \Phi_g^t y = \begin{cases} \sin(t + \arcsin(y)) & , \text{ if } t + \arcsin(y) < \frac{\pi}{2} \\ 1 & , \text{ else, } \end{cases} \quad t > 0, y \in [0, 1].$$

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L.T. : order = 1

Strang : order 2

Theorem 12.5.0.6. Order of simple splitting methods

Die single step methods defined by (12.5.0.3) or (12.5.0.4) are of order ($\rightarrow$ Def. 11.3.2.8) 1 and 2, respectively.

L.T. Strang

§ 12.5.0.7 (Inexact splitting methods)

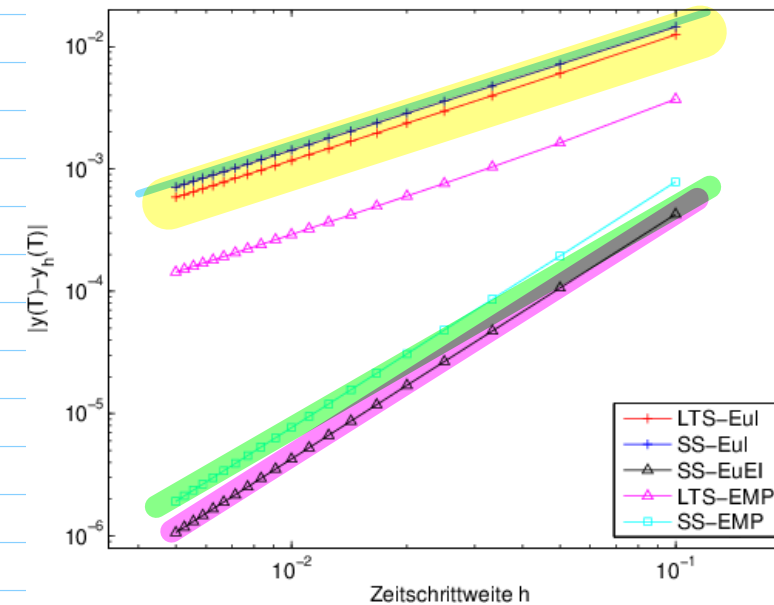


Idea: In (12.5.0.3)/(12.5.0.4) replace

exact evolutions Φ_g^h, Φ_f^h $\rightarrow$ discrete evolutions Ψ_g^h, Ψ_f^h

Exp. 12.5.0.8 (Convergence of inexact splitting methods)

$$\text{IVP : } \dot{y} = \underbrace{\lambda y(1-y)}_{=:f(y)} + \underbrace{\sqrt{1-y^2}}_{=:g(y)}, \quad y(0) = 0.$$



- LTS-Eul explicit Euler method (11.2.1.5) $\rightarrow \Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Lie-Trotter splitting (12.5.0.3)
- SS-Eul explicit Euler method (11.2.1.5) $\rightarrow \Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Strang splitting (12.5.0.4)
- SS-EuEI Strang splitting (12.5.0.4): explicit Euler method (11.2.1.5) $\circ$ exact evolution $\Phi_g^h \circ$ implicit Euler method (11.2.2.2)
- LTS-EMP explicit midpoint method (11.2.3.3) $\rightarrow \Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Lie-Trotter splitting (12.5.0.3)
- SS-EMP explicit midpoint method (11.4.0.9) $\rightarrow \Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Strang splitting (12.5.0.4)

$\triangleright \Psi_g^h, \Psi_f^h$ should match order of splitting scheme

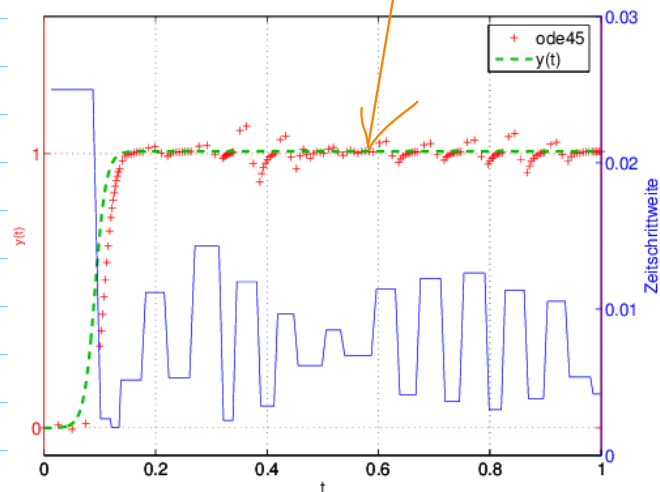
Standard application scenario:

"Splittable" ODEs

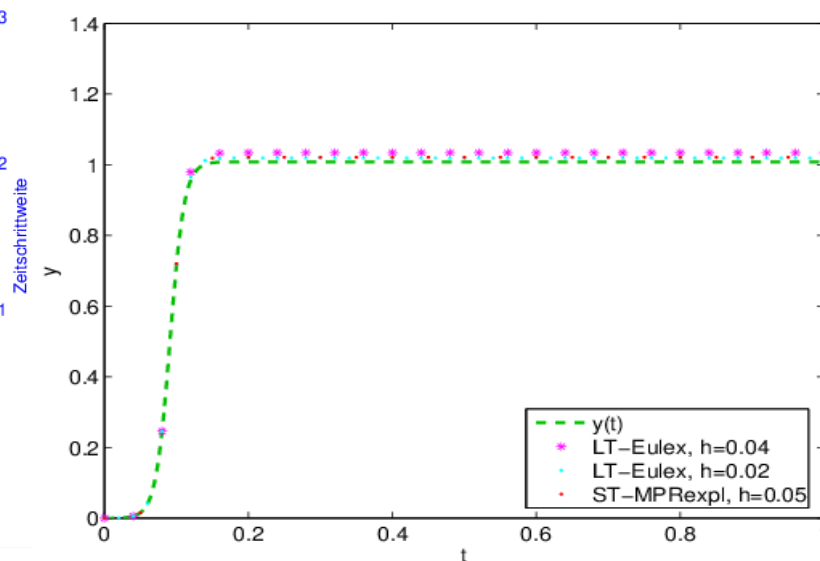
$\dot{y} = f(y) + g(y)$ "difficult" : $\dot{y} = f(y) \rightarrow$ stiff, but with an analytic solution
(e.g., stiff $\rightarrow$ Section 12.2) : $\dot{y} = g(y)$ "easy", amenable to explicit integration.

Exp. 12.5.0.11 (Splitting off stiff components)

IVP: $\dot{y} = \underbrace{\lambda y(1-y)}_{f(y)} + \underbrace{\alpha \sin(y)}_{g(y)}$, $\lambda = 100$, $\alpha = 1$, $y(0) = 10^{-4}$.
 $\lambda \rightarrow$ stiff, α small perturbation



Solution by **ode45**, see Ex. 12.0.0.1



inexact splitting method: solution (y_k)

Ex. 12.5.0.12 (Splitting linear and decoupled terms)

$$\dot{y} = f(y) := -Ay + \begin{bmatrix} g(y_1) \\ \vdots \\ g(y_N) \end{bmatrix}, \quad A = A^T \in \mathbb{R}^{N,N} \text{ positive definite } (\rightarrow \text{Def. 1.1.2.6}), \quad (12.5.0.13)$$

$N \gg 1$

(i) $\lambda_{\min}(A) \approx 1$, $\lambda_{\max}(A) \approx N^2$

(ii) $g: \mathbb{R} \rightarrow \mathbb{R}$, $|g'| \approx 1$

(i) $\Rightarrow \dot{y} = Ay$ will induce **stiff** IVPs

(ii) $\Rightarrow \dot{y}_i = g(y_i) \triangleq$ **non-stiff**

Splitting:

$$\dot{y} = f(y) := -Ay + \begin{bmatrix} g(y_1) \\ \vdots \\ g(y_N) \end{bmatrix}, \quad A = A^T \in \mathbb{R}^{N,N} \text{ positive definite } (\rightarrow \text{Def. 1.1.2.6}), \quad (12.5.0.13)$$

$\downarrow \qquad \qquad \downarrow$
 $f_1(y) \quad f_2(y)$

For: $\dot{y} = f_1(y) \rightarrow$ Use L-stable IRK-SSM (e.g. 2-stage Radau SSM)
 $\hookrightarrow$ A linear ODE!

$\dot{y} = f_2(y) \rightarrow$ Use explicit RK-SSM

⑤

▷ Please study *Review Questions 12.5.0.14* now
(Do not consult these notes or the lecture document)

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