

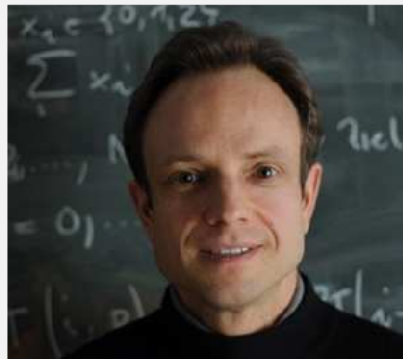
Course Video

Section 1.5.5: Numerical Stability

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Vector and matrix norms

Dependencies.

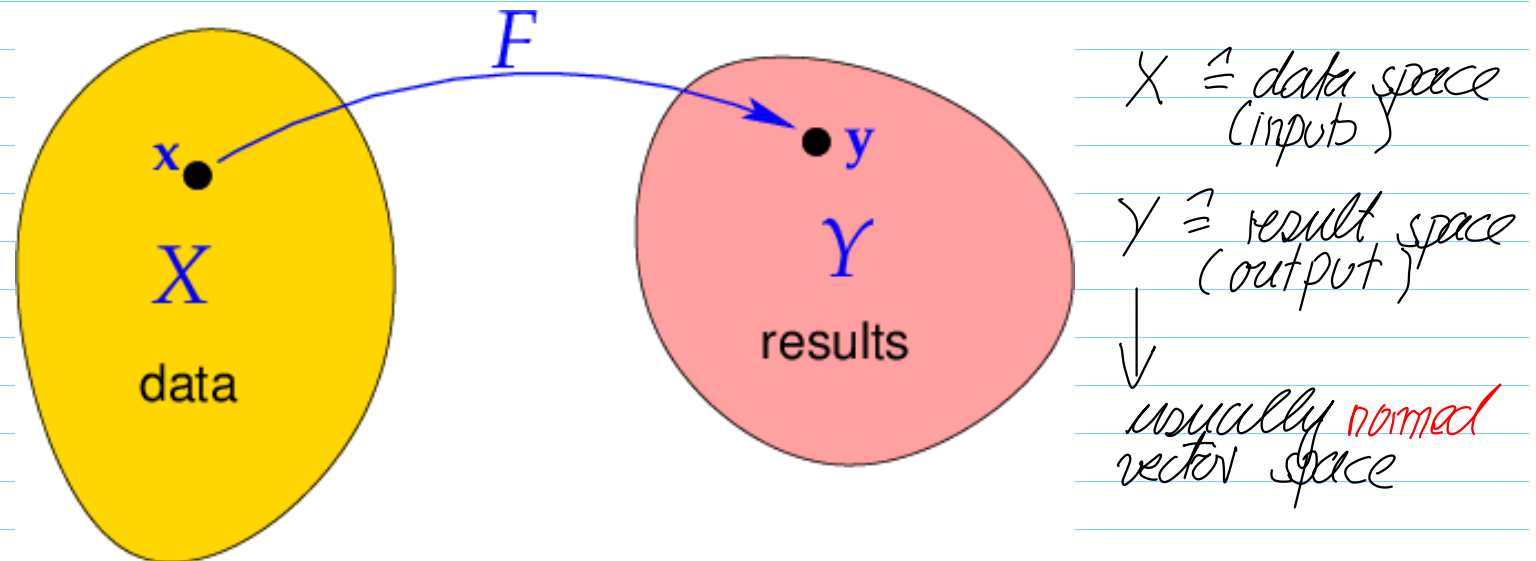
Duration: 35 minutes

I. Computing with Matrices and Vectors

1.5. Machine Arithmetic and Consequences

1.5.5. Numerical Stability (of algorithms)

Problem $\hat{=}$ a function/mapping $F: X \rightarrow Y$



Ex 1.5.5.2: Matrix $\times$ Vector multiplication

$$F: (A, \underline{x}) \in X \mapsto A\underline{x} \in Y$$

$$X = \mathbb{R}^{m,n} \times \mathbb{R}^n \quad Y = \mathbb{R}^m$$

On $\mathbb{R}^n$ we use vector norms, e.g.

$$\|\cdot\|_2 \hat{=}$$
 Euclidean norm, $\|\cdot\|_1$, $\|\cdot\|_\infty$

② Matrix norms induced by vector norms :

Definition 1.5.5.10. Matrix norm

Given vector norms $\|\cdot\|_x$ and $\|\cdot\|_y$ on $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively, the associated **matrix norm** is defined by

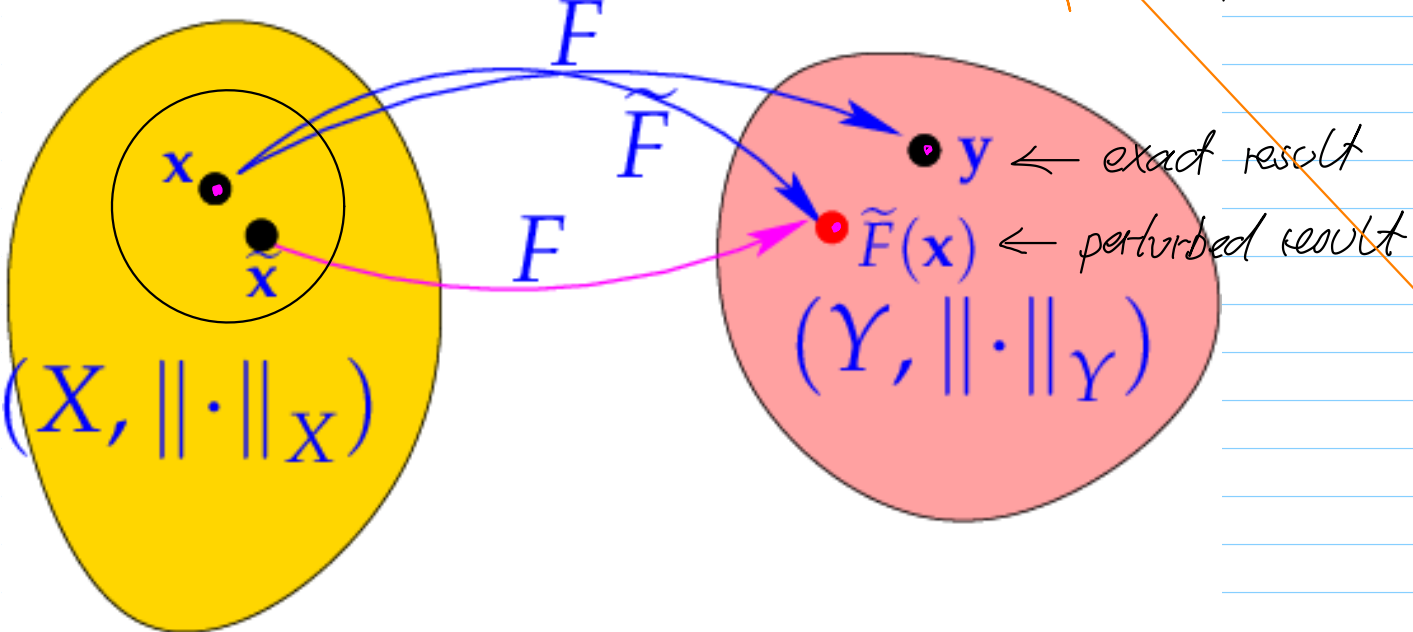
$$\mathbf{M} \in \mathbb{R}^{m,n}: \|\mathbf{M}\| := \sup_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\|\mathbf{M}\mathbf{x}\|_y}{\|\mathbf{x}\|_x}.$$

Definition 1.5.5.19. Stable algorithm

An algorithm $\tilde{F}$ for solving a problem $F: X \mapsto Y$ is **numerically stable** if for all $\mathbf{x} \in X$ its result $\tilde{F}(\mathbf{x})$ (possibly affected by roundoff) is the **exact result** for "slightly perturbed" data:

$$\exists C \approx 1: \forall \mathbf{x} \in X: \exists \tilde{\mathbf{x}} \in X: \|\mathbf{x} - \tilde{\mathbf{x}}\|_X \leq C w(\mathbf{x}) \text{EPS} \|\mathbf{x}\|_X \wedge \tilde{F}(\mathbf{x}) = F(\tilde{\mathbf{x}}).$$

$\tilde{F} \triangleq$ algorithm affected by roundoff #ops.



"Exact data hardly ever available": input $\pi, 1/3$

Sloppily speaking, the impact of roundoff (*) on a *stable algorithm* is of the same order of magnitude as the effect of the inevitable perturbations due to rounding the input data.

Ex. 1.5.5.20 :

Problem : $(A, x) \in \mathbb{R}^{m,n} \times \mathbb{R}^n \mapsto Ax \in \mathbb{R}^m$

$y = \tilde{F}((A, x))$: computed result

[Euclidean vector & matrix norms]

Which perturbations $(\tilde{A}, \tilde{x})$ of data (A, x) yield

$$y = \tilde{A} \tilde{x}$$

Possible choice: $\tilde{x} = x$

$$\tilde{A} = A + \underline{z} x^T, \quad \underline{z} := \frac{y - Ax}{\|x\|_2^2} \in \mathbb{R}^m$$

$$\Rightarrow \tilde{A} \tilde{x} = Ax + \left(\frac{y - Ax}{\|x\|_2^2} \right) x^T x = Ax + y - Ax = y$$

$$F((\tilde{A}, \tilde{x})) = y$$

$$\|A - \tilde{A}\|_2 = \|\underline{z} x^T\|_2 \leq \|x\|_2 \|\underline{z}\|_2 = \frac{\|y - Ax\|_2}{\|x\|_2}$$

$$\text{If } \|y - Ax\|_2 \leq Cmn \cdot \text{EPS} \|x\|_2 \|A\|_2 \quad (*)$$

$$\triangleright \|A - \tilde{A}\|_2 \leq Cmn \cdot \text{EPS} \|A\|_2$$

③ If $(*)$ holds $\forall$ inputs $(A, x) \Rightarrow \hat{F}$ is stable

Rem 1.5.5.22 :

Problem with *sensitive dependence* on data :

$\|F(x) - F(\hat{x})\|_y$ can be large for tiny $\|x - \hat{x}\|_x$

Example :



Tiny perturbations in initial positions & velocities (=data) will lead to huge difference in positions after a long time (=output)

Random output is a stable algorithm !

▷ Review questions 1.5.5.23

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