

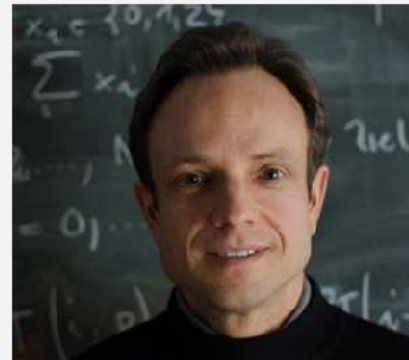
Course Video

Section 3.0.1: Overdetermined Linear Systems of Equations: Examples

Prof. R. Hiptmair, SAM, ETH Zurich

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Prerequisites.

- Elementary linear algebra and geometry

Duration: 20 minutes

III. Direct Methods for Linear Least Squares Problems

Overdetermined LSE :

$$\mathbf{x} \in \mathbb{R}^n: \quad \mathbf{Ax} = \mathbf{b}, \quad (3.0.0.1)$$

$$\mathbf{b} \in \mathbb{R}^m, \quad \mathbf{A} \in \mathbb{R}^{m,n}, \quad m \geq n.$$

$$\begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

↑
tall system matrices

3.0.1 Overdetermined Linear Systems of Equations

Ex 3.0.1.1: Linear parameter estimation in 1D

Law: $y = \alpha x + \beta$ for some unknown parameters $\alpha, \beta \in \mathbb{R}$

Measured: $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, \dots, m$, $m > 2$

$$y_i = \alpha x_i + \beta \quad \forall i \Rightarrow \text{LSE}$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m,2}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^2. \quad (3.0.1.3)$$

$\uparrow$ will be perturbed $\Rightarrow$ no solutions

However: measurement errors have occurred

Ex 3.0.1.4: Linear regression: linear parameter estimation

Law: $y = \underline{a}^T \underline{x} + \beta$, parameters $\underline{a} \in \mathbb{R}^n$, $\beta \in \mathbb{R}$
 $\underline{x} \in \mathbb{R}^n$

Measurements: $(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}$, $i = 1, \dots, m$

$$y_i = \underline{a}^T \underline{x}_i + \beta$$

OD, if $m > n+1$

$$\begin{bmatrix} x_1^T & 1 \\ \vdots & \vdots \\ x_m^T & 1 \end{bmatrix} \begin{bmatrix} \underline{a} \\ \beta \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m,n+1}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^{n+1}, \quad (3.0.1.5)$$

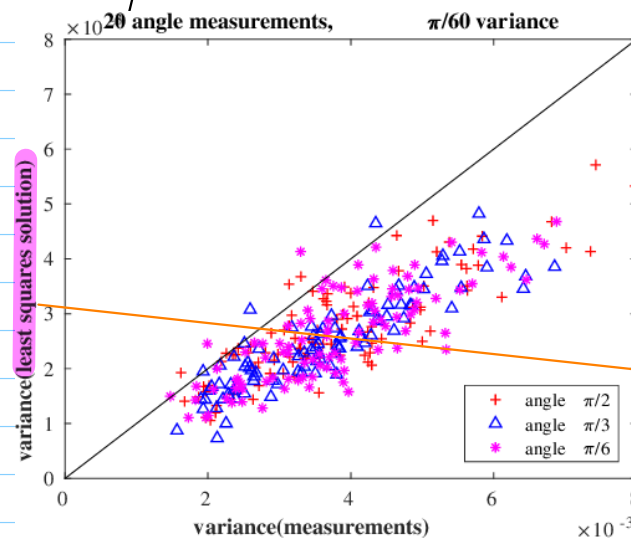
$\uparrow$ maybe perturbation

Ex 3.0.1.6: Measuring the angles of a triangle

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \\ \pi \end{bmatrix} \quad \text{---} \hat{=} \text{measured angles} \quad (3.0.1.7)$$

$\Sigma \text{ angles} = \pi$

Principle in data science: You cannot afford not to use every piece of information available



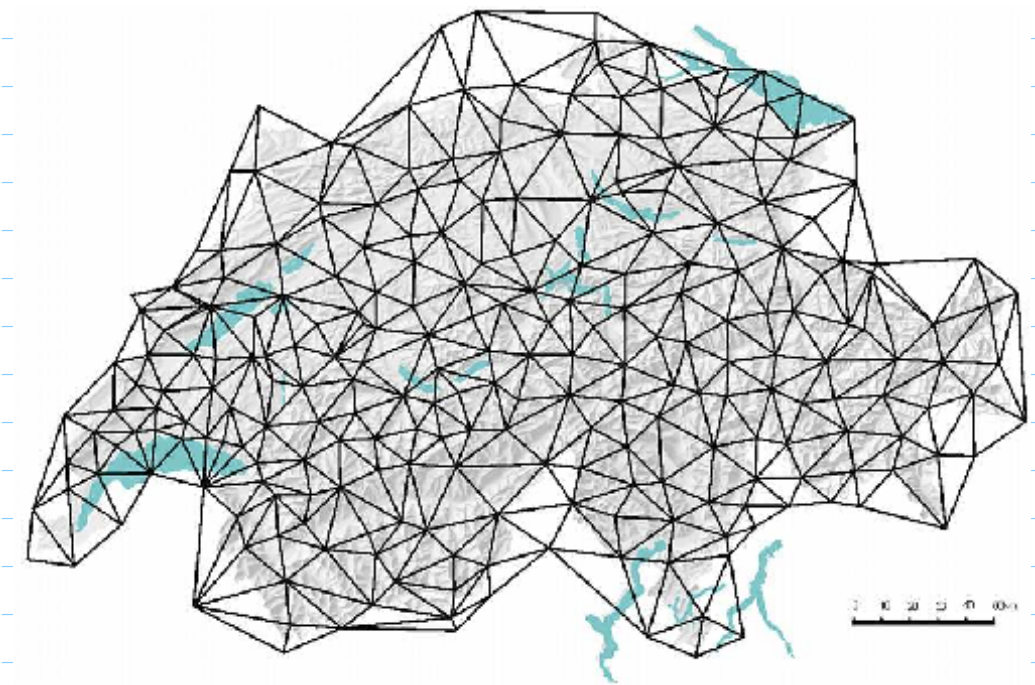
$\triangleleft$ Randomly perturbed angles

$\triangleright$ Variance reduction

$\rightarrow$ this chapter

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Ex 3.0.1.8 : Angles in triangulations



Measured:
angles

1. each angle is supposed to be equal to its measured value,
2. the sum of interior angles is π for every triangle,
3. the sum of the angles at an interior node is 2π .

+ #triag. extra equ.
+ #int. nodes extra equ. $\Rightarrow$ OD LSE

Ex. 3.0.1.9. Point locations from distances

1D (on a line) : $x_i \in \mathbb{R}, i=1, \dots, n, n \geq 2$
[Assume $x_i < x_{i+1}$]

Measure : $d_{ij} = |x_i - x_j|, i \neq j$

Use all data $\Rightarrow$ OD LSE

$$\begin{matrix} x_i - x_j = d_{ij}, \\ 1 \leq j < i \leq n. \end{matrix} \quad \Leftrightarrow \quad \begin{bmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ -1 & 0 & 1 & 0 & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ -1 & \dots & & & 0 & 1 \\ 0 & -1 & 1 & & 0 & \\ \vdots & & & & \vdots & \\ \vdots & & & & \vdots & \\ 0 & \dots & & & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_{12} \\ d_{13} \\ \vdots \\ d_{1n} \\ d_{23} \\ \vdots \\ d_{n-1,n} \end{bmatrix} \quad (3.0.1.10)$$

$\triangleright \begin{pmatrix} n \\ 2 \end{pmatrix}$ equ. for n unknowns

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Review questions:

Please answer review questions 3.0.1.11
without aids.

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