

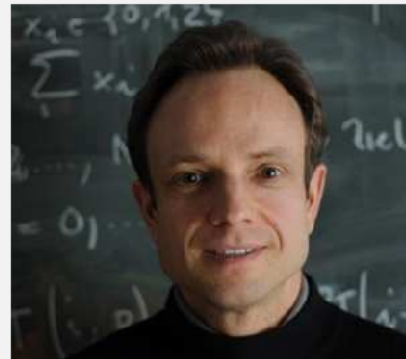
Course Video

Section 3.3: Orthogonal Transformation Methods

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- From LA: orthogonal matrices

Dependencies. Section 3.1.1

Duration: 15 minutes

III. Direct Methods for Linear Least Squares Problems

3.3. Orthogonal Transformation Methods

Setting: OD LSE $Ax = \underline{b}$, $A \in \mathbb{R}^{m,n}$, $m \geq n$

Assume: FRC: $\Leftrightarrow \text{rank}(A) = n$

LSQ sol.: $\{x\} = \text{lsq}(A, \underline{b}) = \underset{y \in \mathbb{R}^n}{\text{argmin}} \| \underline{b} - Ay \|_2^2$

3.3.1. Orthogonal Transformation Methods: Idea

As GE: Want to transform $Ax = \underline{b} \rightarrow \tilde{A}x = \tilde{\underline{b}}$

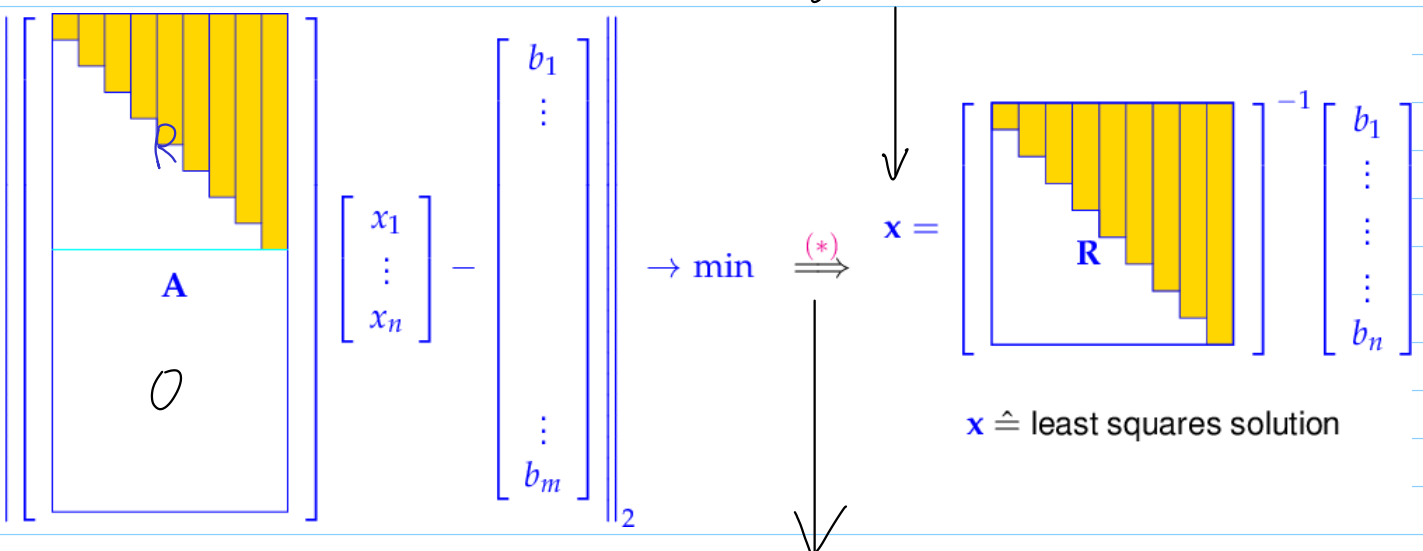
s.t. (i) LSQ sol. of $\tilde{A}x = \tilde{\underline{b}}$ "easy to compute"

(ii) $\text{lsq}(A, \underline{b}) = \text{lsq}(\tilde{A}, \tilde{\underline{b}})$

(i) for GE: triangular system

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(i) A triangular



$\text{rank}(A) = n$
 $\rightarrow R$ regular
 Make first n component of $Ax - b$ vanish

(ii) Idea: $\|\cdot\|_2$ -preserving linear trf.



Idea: If we have a (transformation) matrix $T \in \mathbb{R}^{m,m}$ satisfying

$$\|Ty\|_2 = \|y\|_2 \quad \forall y \in \mathbb{R}^m, \quad (3.3.1.3)$$

then $\argmin_{y \in \mathbb{R}^n} \|Ay - b\|_2 = \argmin_{y \in \mathbb{R}^n} \|\tilde{A}y - \tilde{b}\|_2 = \text{lsq}(A, b)$

where $\tilde{A} = TA$ and $\tilde{b} = Tb$.

3.3.2. Orthogonal Matrices

$$\|Tx\|_2 = \|x\|_2 \quad \forall x \in \mathbb{R}^n \quad (*)$$

$$x^T y = \frac{1}{2} (\|x+y\|_2^2 - \|x\|_2^2 - \|y\|_2^2) \quad [\text{Polarization identity}]$$

$$(Tx)^T (Ty) = x^T T^T Ty \stackrel{(*)}{=} x^T y \quad \forall x, y \in \mathbb{R}^n$$

$$T^T T = I$$

Definition 3.3.2.1. Unitary and orthogonal matrices $\rightarrow$

- $Q \in \mathbb{K}^{n,n}, n \in \mathbb{N}$, is **unitary**, if $Q^{-1} = Q^H$.
- $Q \in \mathbb{R}^{n,n}, n \in \mathbb{N}$, is **orthogonal**, if $Q^{-1} = Q^T$. $\Leftrightarrow \|Qx\|_2 = \|x\|_2 \quad \forall x \in \mathbb{R}^n$
 $\rightarrow$ Cols/rows form an ONB of $\mathbb{R}^n$

③

Review questions

▷ Answer Review Questions 3.3.2.3
without aids

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