

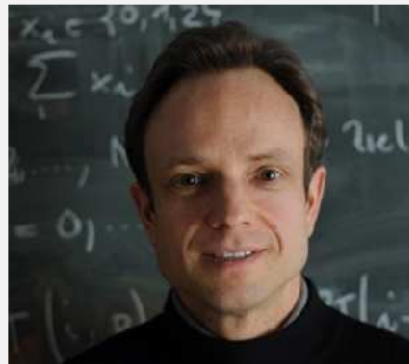
Course Video

Section 3.3.3.1: QR-Decomposition: Theory

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Tablet notes may have been corrected and/or extended after the recording of the video!

Prerequisites.

- From LA: Gram-Schmidt orthogonalization

Duration: 20 minutes

III. Direct Methods for Linear Least Squares Problems

3.3. Orthogonal Transformation Methods

3.3.3. QR-Decomposition

3.3.3.1. QR-Decomposition: Theory

Recall: Gram-Schmidt orthonormalization of vectors (from LA): $\{\underline{a}^1, \dots, \underline{a}^n\} \subset \mathbb{R}^m$, $m \geq n$, linearly independent

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1:  $\mathbf{q}^1 := \frac{\mathbf{a}^1}{\|\mathbf{a}^1\|_2}$ ; % 1st output vector
2: for  $j = 2, \dots, n$  do
3:   { % Orthogonal projection
4:      $\mathbf{q}^j := \mathbf{a}^j$ ;
5:     for  $\ell = 1, 2, \dots, j-1$  do (GS)
6:       {  $\mathbf{q}^j \leftarrow \mathbf{q}^j - \langle \mathbf{a}^j, \mathbf{q}^\ell \rangle \mathbf{q}^\ell$ ; }
7:     if ( $\mathbf{q}^j = 0$ ) then STOP
8:     else {  $\mathbf{q}^j \leftarrow \frac{\mathbf{q}^j}{\|\mathbf{q}^j\|_2}$ ; }
9:   }
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$$(\mathbf{q}^i)^T \mathbf{q}^j = \delta_{ij}$$

Theorem 3.3.3.2. Span property of G.S. vectors

If $\{\mathbf{a}^1, \dots, \mathbf{a}^n\} \subset \mathbb{R}^m$ is linearly independent, then Algorithm (GS) computes orthonormal vectors $\mathbf{q}^1, \dots, \mathbf{q}^n \in \mathbb{R}^m$ satisfying

$$\text{Span}\{\mathbf{q}^1, \dots, \mathbf{q}^\ell\} = \text{Span}\{\mathbf{a}^1, \dots, \mathbf{a}^\ell\}, \quad (1.5.1.2)$$

for all $\ell \in \{1, \dots, n\}$.

②

$$Q := [q^1, \dots, q^n] \in \mathbb{R}^{m,n}, \quad Q^T Q = I_n$$

$$A := [a^1, \dots, a^n] \in \mathbb{R}^{m,n}$$

(1.5.1.2): $\text{Span}\{q^1, \dots, q^n\} = \text{Span}\{a^1, \dots, a^n\} \quad \forall l=1, \dots, n$

$$\Rightarrow \exists t_{ik} \in \mathbb{R} : q^l = \sum_{i=1}^l t_{li} a^i, \quad l=1, \dots, n$$

$$\begin{aligned} q^1 &= t_{11} a^1 \\ q^2 &= t_{12} a^1 + t_{22} a^2 \\ q^3 &= t_{13} a^1 + t_{23} a^2 + t_{33} a^3 \\ &\vdots \\ q^n &= t_{1n} a^1 + t_{2n} a^2 + \dots + t_{nn} a^n. \end{aligned}$$

$$(T)_{ij} := \begin{cases} t_{ij}, & \text{if } i \leq j \\ 0, & \text{else.} \end{cases}$$

$\exists T \in \mathbb{R}^{n,n}$ upper triangular: $Q = AT$, (3.3.3.3)

regular, since $\text{rank}(A) = n = \text{rank}(Q)$
 $(t_{ii} \neq 0) \Rightarrow \text{rank}(T) = n$

$$\Rightarrow A = Q T^{-1} = QR$$

Lemma 1.3.1.9. Group of regular diagonal/triangular matrices

$A, B \begin{cases} \text{diagonal} \\ \text{upper triangular} \\ \text{lower triangular} \end{cases} \Rightarrow AB \text{ and } A^{-1} \begin{cases} \text{diagonal} \\ \text{upper triangular} \\ \text{lower triangular} \end{cases}$
 (assumes that A is regular)

$$\begin{bmatrix} \text{yellow staircase} \\ T \end{bmatrix}^{-1} = \begin{bmatrix} \text{yellow staircase} \\ R \end{bmatrix}$$



$$A = QR \Leftrightarrow$$

$$\begin{bmatrix} A \\ \in \mathbb{R}^{m,n} \\ \text{full rank} \end{bmatrix} = \begin{bmatrix} Q \\ \in \mathbb{R}^{m,n} \\ \text{orthonormal columns} \\ Q^T Q = I_n \end{bmatrix} \begin{bmatrix} R \\ \in \mathbb{R}^{n,n} \\ \text{regular} \end{bmatrix}$$

$\hat{=}$ economical / thin QR-decomposition

Full QR-decomposition: extend $\{q^1, \dots, q^n\}$ to an ONB of $\mathbb{R}^m$

$$\blacktriangleright A = \tilde{Q} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} Q & \tilde{Q} \\ q^1 & q^n & q^{n+1} & \dots & q^m \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$\tilde{Q}$ is orthogonal matrix

$$\Leftrightarrow \tilde{Q}^T A = \begin{bmatrix} R \\ 0 \end{bmatrix}$$

③

Theorem 3.3.3.4. QR-decomposition $\rightarrow$,

For any matrix $\mathbf{A} \in \mathbb{K}^{n,k}$ with $\text{rank}(\mathbf{A}) = k$ there exists

- (i) a unique Matrix $\mathbf{Q}_0 \in \mathbb{R}^{n,k}$ that satisfies $\mathbf{Q}_0^H \mathbf{Q}_0 = \mathbf{I}_k$, and a unique upper triangular Matrix $\mathbf{R}_0 \in \mathbb{K}^{k,k}$ with $(\mathbf{R})_{i,i} > 0, i \in \{1, \dots, k\}$, such that

$$\mathbf{A} = \mathbf{Q}_0 \cdot \mathbf{R}_0 \quad (\text{"economical" QR-decomposition}),$$

- (ii) a unitary Matrix $\mathbf{Q} \in \mathbb{K}^{n,n}$ and a unique upper triangular $\mathbf{R} \in \mathbb{K}^{n,k}$ with $(\mathbf{R})_{i,i} > 0, i \in \{1, \dots, n\}$, such that

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{R} \quad (\text{full QR-decomposition}).$$

If $\mathbb{K} = \mathbb{R}$ all matrices will be real and $\mathbf{Q}$ is then orthogonal.

$\triangleright$ Answer review questions 3.3.3.8 !

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