

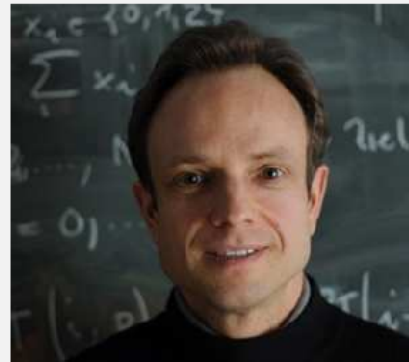
Course Video

Section 3.3.3.1: QR-Decomposition: Theory

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- From LA: Gram-Schmidt orthogonalization

Duration: 18 minutes

III. Direct Methods for Linear Least Squares Problems

3.3. Orthogonal Transformation Methods

3.3.3. QR-Decomposition

3.3.3.1. QR-Decomposition: Theory

Recall : Gram-Schmidt orthonormalization of $\{\mathbf{a}^1, \dots, \mathbf{a}^n\} \subset \mathbb{R}^m$, $m \geq n$, lin. indep.

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1:  $\mathbf{q}^1 := \frac{\mathbf{a}^1}{\|\mathbf{a}^1\|_2}$ ; % 1st output vector
2: for  $j = 2, \dots, n$  do
  { % Orthogonal projection
3:    $\mathbf{q}^j := \mathbf{a}^j$ ;
4:   for  $\ell = 1, 2, \dots, j-1$  do (GS)
5:     {  $\mathbf{q}^j \leftarrow \mathbf{q}^j - \langle \mathbf{a}^j, \mathbf{q}^\ell \rangle \mathbf{q}^\ell$ ; }
6:     if ( $\mathbf{q}^j = \mathbf{0}$ ) then STOP
7:     else {  $\mathbf{q}^j \leftarrow \frac{\mathbf{q}^j}{\|\mathbf{q}^j\|_2}$ ; }
8:   }
```

Theorem 3.3.3.2. Span property of G.S. vectors

If $\{\mathbf{a}^1, \dots, \mathbf{a}^n\} \subset \mathbb{R}^m$ is linearly independent, then Algorithm (GS) computes **orthonormal vectors** $\mathbf{q}^1, \dots, \mathbf{q}^k \in \mathbb{R}^m$ satisfying

$$\text{Span}\{\mathbf{q}^1, \dots, \mathbf{q}^\ell\} = \text{Span}\{\mathbf{a}^1, \dots, \mathbf{a}^\ell\}, \quad (1.5.1.2)$$

for all $\ell \in \{1, \dots, n\}$.

②

$Q = [q^1, \dots, q^n] \in \mathbb{R}^{m,n}, \quad Q^T Q = I$

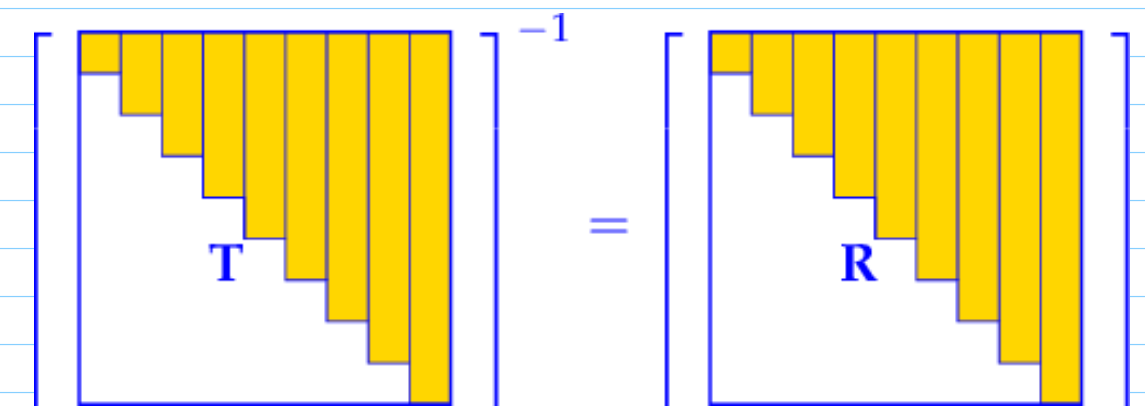
$A = [a^1, \dots, a^n] \in \mathbb{R}^{m,n}$

$q^1 = t_{11}a^1$
 $q^2 = t_{12}a^1 + t_{22}a^2$
 $q^3 = t_{13}a^1 + t_{23}a^2 + t_{33}a^3$
 $\vdots$
 $q^n = t_{1n}a^1 + t_{2n}a^2 + \dots + t_{nn}a^n$

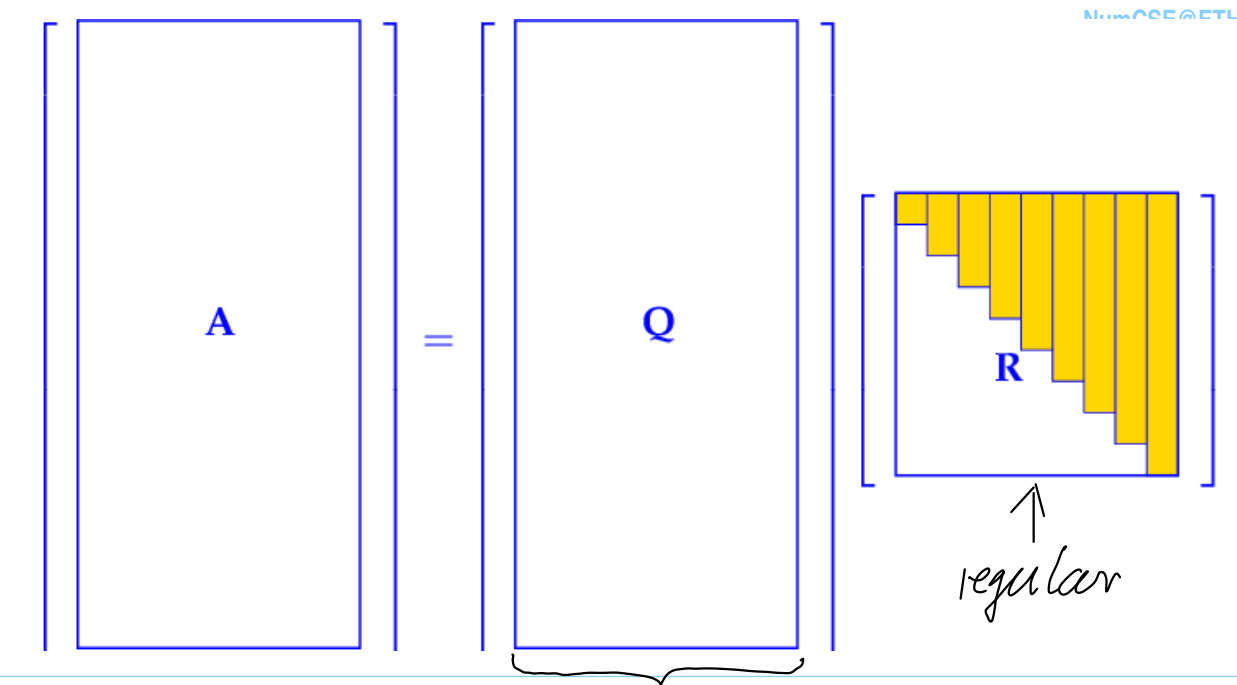
$(T)_{ij} := \begin{cases} t_{ij}, & \text{if } i \leq j \\ 0, & \text{else} \end{cases}$

$\exists T \in \mathbb{R}^{n,n}$ upper triangular: $Q = AT$, (3.3.3.3)
 $\uparrow$ regular $\Rightarrow \text{rank}(Q) = n \Rightarrow \text{rank}(T) = n$

∇ T^{-1} is also upper triangular



$A = QR \Leftrightarrow$
 $\uparrow$ full rank

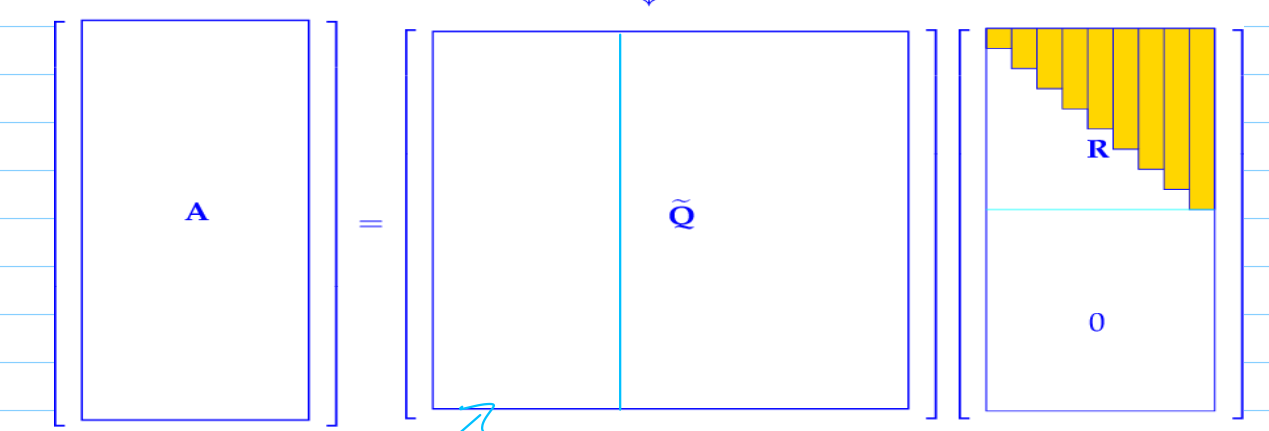


orthonormal columns!

$\hat{=}$ economical QR-decomposition

• Full QR-decomposition: extend $\{q^1, \dots, q^n\}$ to an ONB of $\mathbb{R}^m \rightarrow$ columns of orthogonal $\tilde{Q} \in \mathbb{R}^{m,m}$

$\blacktriangleright A = \tilde{Q} \begin{bmatrix} R \\ 0 \end{bmatrix}$
 $\updownarrow$



$\Leftrightarrow \tilde{Q}^T A = \begin{bmatrix} R \\ 0 \end{bmatrix}$

③

Summary :

Theorem 3.3.3.4. QR-decomposition $\rightarrow$,

For any matrix $\mathbf{A} \in \mathbb{K}^{n,k}$ with $\text{rank}(\mathbf{A}) = k$ there exists

- (i) a unique Matrix $\mathbf{Q}_0 \in \mathbb{R}^{n,k}$ that satisfies $\mathbf{Q}_0^H \mathbf{Q}_0 = \mathbf{I}_k$, and a unique *upper triangular* Matrix $\mathbf{R}_0 \in \mathbb{K}^{k,k}$ with $(\mathbf{R})_{i,i} > 0, i \in \{1, \dots, k\}$, such that

$$\mathbf{A} = \mathbf{Q}_0 \cdot \mathbf{R}_0 \quad (\text{"economical" QR-decomposition}) ,$$

- (ii) a *unitary* Matrix $\mathbf{Q} \in \mathbb{K}^{n,n}$ and a unique *upper triangular* $\mathbf{R} \in \mathbb{K}^{n,k}$ with $(\mathbf{R})_{i,i} > 0, i \in \{1, \dots, n\}$, such that

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{R} \quad (\text{full QR-decomposition}) .$$

If $\mathbb{K} = \mathbb{R}$ all matrices will be real and $\mathbf{Q}$ is then *orthogonal*.

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