

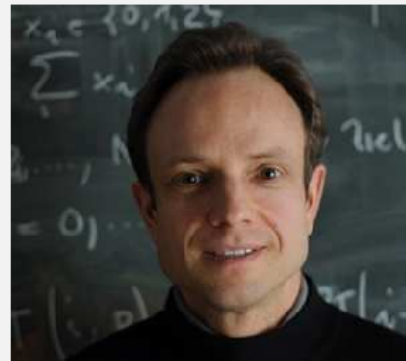
Course Video

Section 3.3.5: Modification Techniques for QR-Decomposition

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependencies. Section 3.3.3.2, Section 2.6

Duration: 45 minutes

III. Direct Methods for Linear Least Squares Problems

3.3. Orthogonal Transformation Methods

3.3.5. Modification Techniques for QR-Decomposition

Given: • $A \in \mathbb{R}^{m,n}$, $m \geq n$, $\text{rank}(A) = n$

• QR-dec (Full) $A = QR$, $Q \in \mathbb{R}^{m,m}$ orthogonal, $R \in \mathbb{R}^{m,n}$ upper triangular

↑
stored as a product of OTs *

Task: Efficiently compute $\tilde{A} = \tilde{Q}\tilde{R}$,
 $\tilde{A} \approx$ "slight modification" of A

* product of $n-1$ Householder matrices

②

3.3.5.1 : Rank-1 Modification

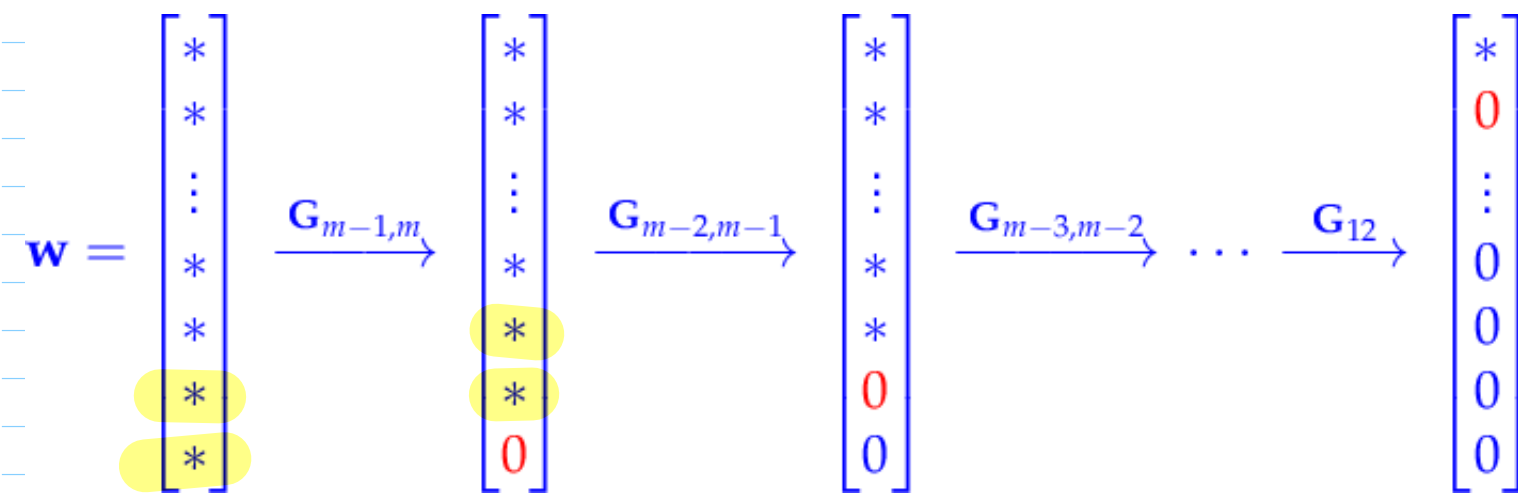
$$\tilde{A} = A + \underline{u}\underline{v}^T, \underline{u} \in \mathbb{R}^m, \underline{v} \in \mathbb{R}^n$$

$$\hat{A} = QR + \underline{u}\underline{v}^T = Q(R + \underbrace{Q^T \underline{u}}_{=: \underline{w}} \underline{v}^T)$$

Step ① : Compute $\underline{w} := Q^T \underline{u}$

→ Cost = $O(mn)$ for $m, n \rightarrow \infty$

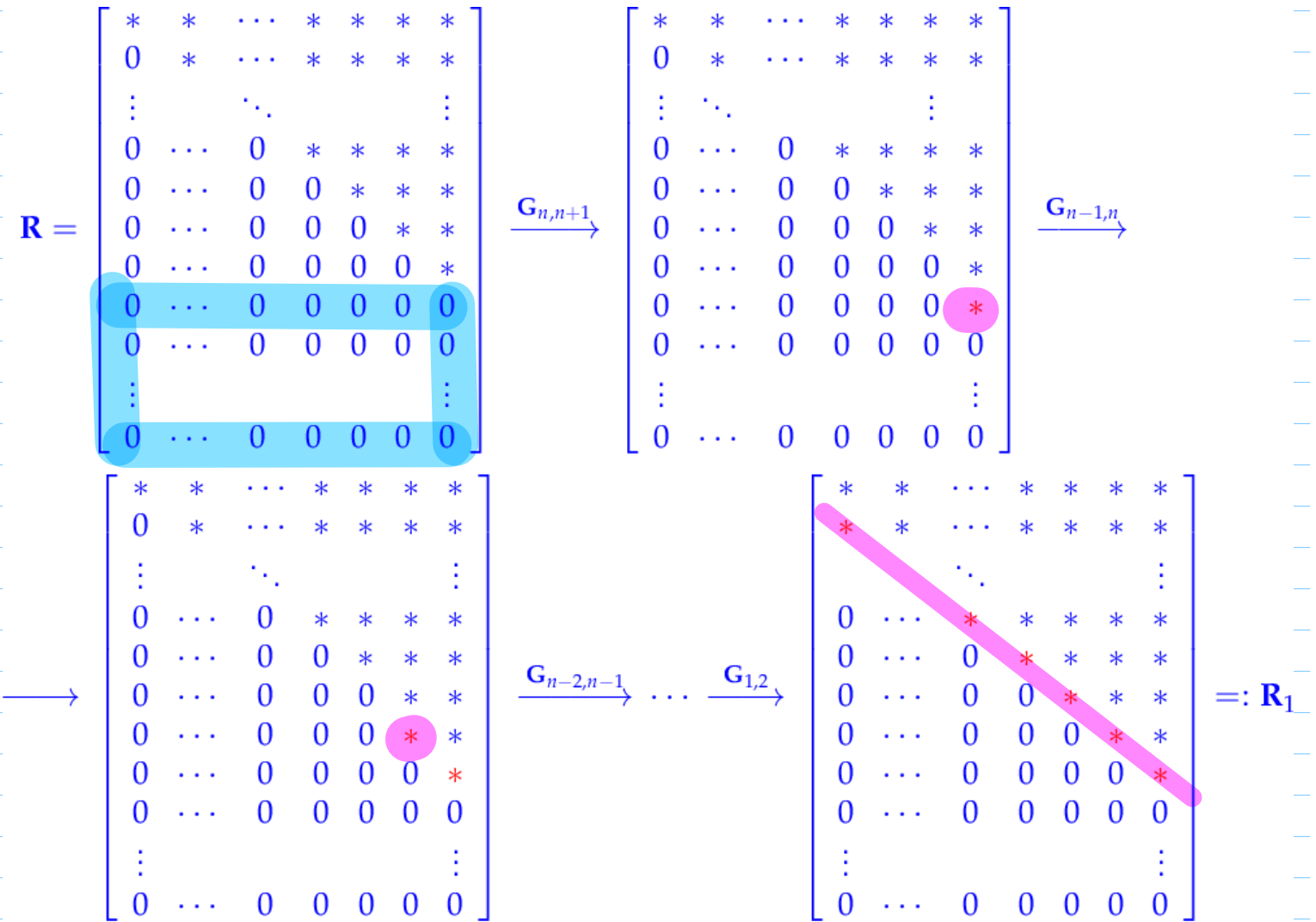
Step ② : $\underline{w} \rightarrow \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$ by successive Givens rotations



→ Cost = $O(m)$

$\uparrow$
 $\|\underline{w}\|_2 \approx 1$

Action on R



$R_1 + \|\underline{w}\|_2 \underline{e}_1 \underline{v}^T$ will have the same structure as R ,
 $\uparrow$
 attack with Givens rotations

③

$$\mathbf{R}_1 + \|\mathbf{w}\|_2 \mathbf{e}_1 \mathbf{v}^\top \xrightarrow{\mathbf{G}_{12}} \begin{bmatrix} * & * & \cdots & * & * & * & * \\ 0 & * & \cdots & * & * & * & * \\ & & \ddots & & & & \\ 0 & \cdots & * & * & * & * & * \\ 0 & \cdots & 0 & * & * & * & * \\ 0 & \cdots & 0 & 0 & * & * & * \\ 0 & \cdots & 0 & 0 & 0 & * & * \end{bmatrix} \xrightarrow{\mathbf{G}_{23}} \cdots$$

$$\xrightarrow{\mathbf{G}_{n-2,n-1}} \begin{bmatrix} * & * & \cdots & * & * & * & * \\ 0 & * & \cdots & * & * & * & * \\ & & \ddots & & & & \\ 0 & \cdots & 0 & * & * & * & * \\ 0 & \cdots & 0 & 0 & * & * & * \\ 0 & \cdots & 0 & 0 & 0 & * & * \\ 0 & \cdots & 0 & 0 & 0 & * & * \end{bmatrix} \xrightarrow{\mathbf{G}_{n-1,n}} \begin{bmatrix} * & * & \cdots & * & * & * & * \\ 0 & * & \cdots & * & * & * & * \\ & & \ddots & & & & \\ 0 & \cdots & 0 & * & * & * & * \\ 0 & \cdots & 0 & 0 & * & * & * \\ 0 & \cdots & 0 & 0 & 0 & * & * \\ 0 & \cdots & 0 & 0 & 0 & 0 & * \end{bmatrix} =: \tilde{\mathbf{R}}.$$

$$(\mathbf{G}_{n-1,n} \mathbf{G}_{n-2,n-1} \cdots \mathbf{G}_{23} \mathbf{G}_{12})(\mathbf{R}_1 + \|\mathbf{w}\|_2 \mathbf{e}_1 \mathbf{v}^\top) = \tilde{\mathbf{R}} \quad (\text{upper triangular!}). \quad (3.3.5.3)$$

$\underbrace{\mathbf{G}_{n-1,n} \mathbf{G}_{n-2,n-1} \cdots \mathbf{G}_{23} \mathbf{G}_{12}}_{=: \mathbf{Q}_2} \rightarrow (\text{cost} \approx O(n^2) \text{ for } n \rightarrow \infty)$

$\triangleright \tilde{\mathbf{A}} = \mathbf{Q} \mathbf{Q}_1^\top \mathbf{Q}_2^\top \tilde{\mathbf{R}}$

Total cost = $O(mn + n^2)$ for large m, n
 $\ll O(mn^2)$, if both m, n large

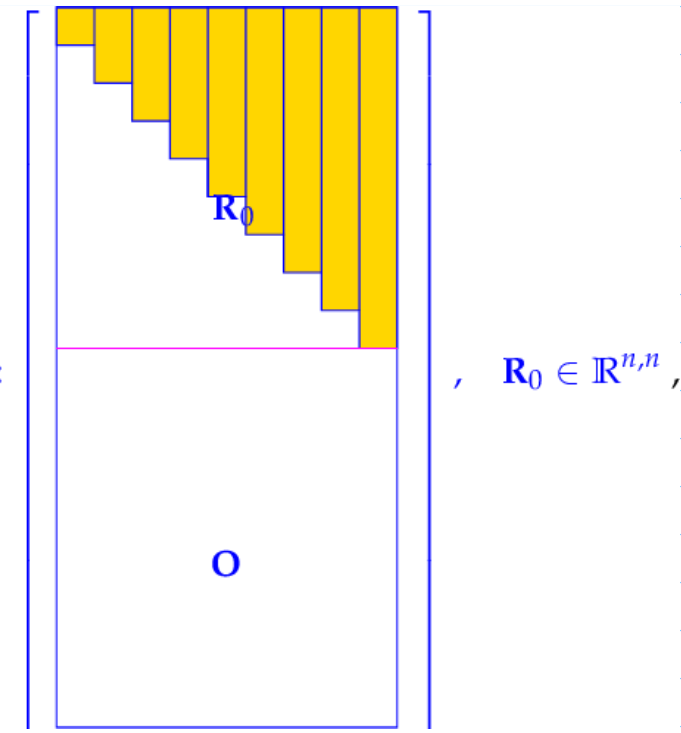
3.3.5.2. Adding a column

$$\mathbf{A} \in \mathbb{R}^{m,n} \rightarrow \tilde{\mathbf{A}} = [\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{v}, \mathbf{a}_k, \dots, \mathbf{a}_n] \in \mathbb{R}^{m,n+1}, \quad \mathbf{v} \in \mathbb{R}^m, \quad \mathbf{a}_j := (\mathbf{A})_{:,j}. \quad (3.3.5.4)$$

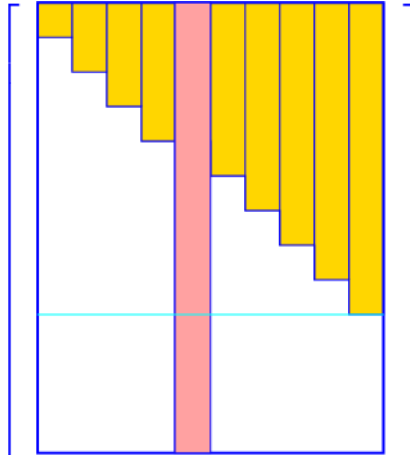
Assume: $m \geq n+1$

full QR-dec.

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \Leftrightarrow \mathbf{Q}^\top \mathbf{A} = [\mathbf{Q}^\top \mathbf{a}_1, \dots, \mathbf{Q}^\top \mathbf{a}_n] = \mathbf{R} =:$$



(4)

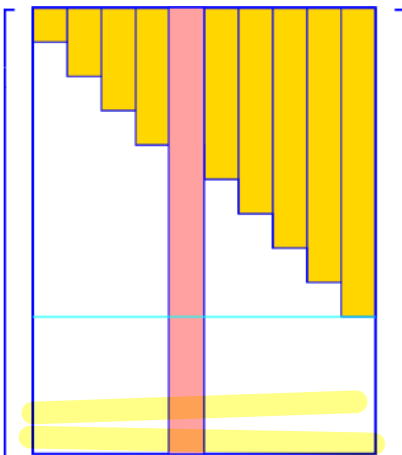
$$Q^T \tilde{A} = [Q^T a_1, \dots, Q^T a_{k-1}, Q^T v, Q^T a_k, \dots, Q^T a_n] =$$


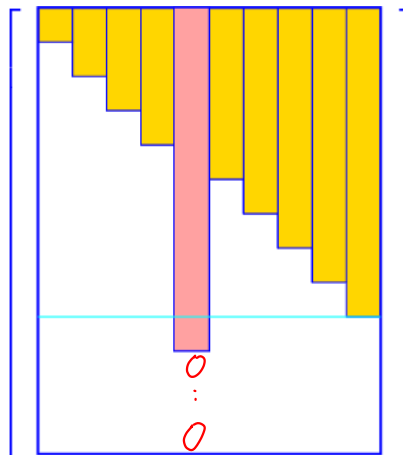
$$=: W \in \mathbb{R}^{m, n+1},$$

↑
k-th column: $Q^T v \in \mathbb{R}^m$

① Compute $\underline{w} := Q^T v$: Cost = $O(mn)$

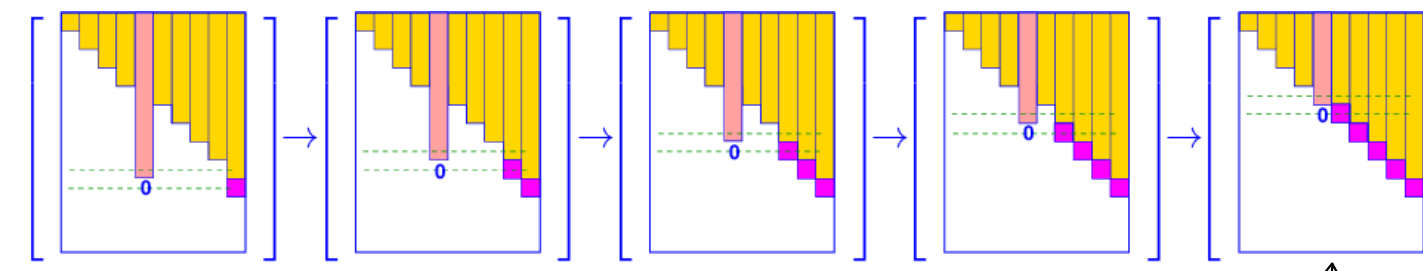
② $m \cdot n - 1$ Givens rot. $\Rightarrow$ last $m \cdot n - 1$ rows of $W \rightarrow \underline{0}$

$$W =$$


$$\xrightarrow{G_{m-1,m}} \dots \xrightarrow{G_{n+1,n+2}}$$


$$=: T.$$

③ $T \rightarrow$ upper triangular $\hat{R} \in \mathbb{R}^{m, n+1}$ by $n-k$ Givens rotations (bottom up)



--- $\triangleq$ target rows of Givens rotations, $\blacksquare \triangleq$ new entries $\neq 0$

$\uparrow$
 $\hat{R}$

$$\text{Cost} = O((n-k)^2)$$

$$\text{Total cost} = O(mn + (n-k)^2) \text{ for } m, n \rightarrow \infty$$

3.3.5.3. Adding a row

$$A \in \mathbb{R}^{m,n} \mapsto \tilde{A} = \begin{bmatrix} (A)_{1,:} \\ \vdots \\ (A)_{k-1,:} \\ v^T \\ (A)_{k,:} \\ \vdots \\ (A)_{m,:} \end{bmatrix}, \text{ with given } v \in \mathbb{R}^n. \quad (3.3.5.5)$$

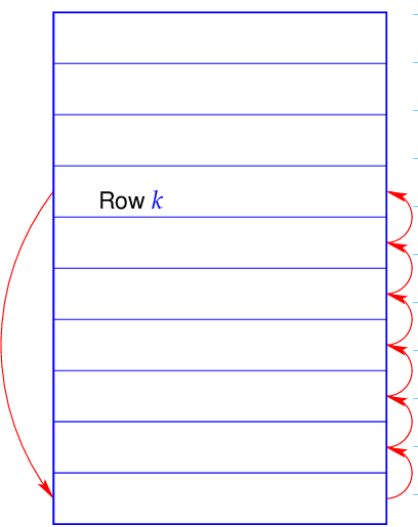
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① Move new row to the bottom

by a cyclic permutation of rows of $\tilde{A}$

$\hat{=}$ orth. trf. $P \in \mathbb{R}^{m+1, m+1}$

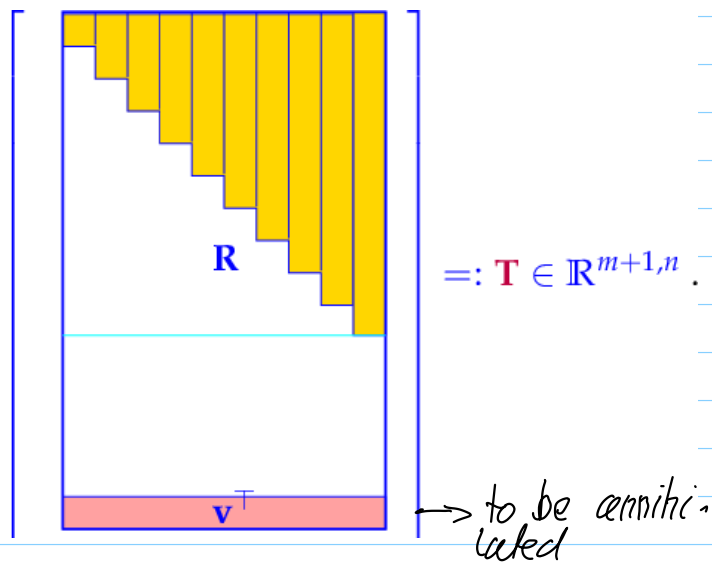
Just reindexing : no cost



$$Q^T A = R$$

$$P\tilde{A} = \begin{bmatrix} A \\ v^T \end{bmatrix} \rightarrow \begin{bmatrix} Q^T & 0 \\ 0 & 1 \end{bmatrix} P\tilde{A} = \begin{bmatrix} R \\ v^T \end{bmatrix} =$$

$\uparrow$
 $\in \mathbb{R}^{(m+1), (m+1)}$ orthogonal



Idea: Use n suitable Givens rotations

$$T = \begin{bmatrix} * & \dots & \dots & * \\ 0 & * & & \vdots \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & \dots & \dots & * & * & * \end{bmatrix} \xrightarrow{G_{1,m+1}} \begin{bmatrix} * & \dots & \dots & * \\ 0 & * & & \vdots \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & * & \dots & * & * & * \end{bmatrix} \xrightarrow{G_{2,m+1}} \dots$$

$$\dots \xrightarrow{G_{m-1,m+1}} \begin{bmatrix} * & \dots & \dots & * \\ 0 & * & & \vdots \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \xrightarrow{G_{m,m+1}} \begin{bmatrix} * & \dots & \dots & * \\ 0 & * & & \vdots \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} := \tilde{R} \quad (3.3.5.6)$$

$$\text{Cost} = O(n^2)$$

⑥

Review questions:

▷ Please answer review questions 3.3.5.7

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