

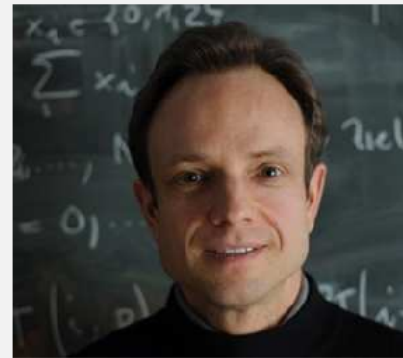
Course Video

Section 4.1.2: LT-FIR Linear Mappings

Prof. R. Hiptmair, SAM, ETH Zurich

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Dependencies. Section 4.1.1

Duration: 24 minutes

IV. Filtering Algorithms

4.1. Filters and Convolutions

4.1.2. LT-FIR Linear Mappings

Definition 4.1.1.2. Finite channel/filter

A channel/filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **finite**, if every input signal of finite duration produces an output signal of finite duration,

$$\{ \exists M \in \mathbb{N}: |j| > M \Rightarrow x_j = 0 \} \Rightarrow \exists N \in \mathbb{N}: |k| > N \Rightarrow (F((x_j)_{j \in \mathbb{Z}}))_k = 0, \quad (4.1.1.3)$$

Definition 4.1.1.5. Time-invariant channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **time-invariant (TI)**, if shifting the input in time leads to the same output shifted in time by the same amount; it *commutes with the time shift operator* from (4.1.1.4):

$$\forall (x_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \forall m \in \mathbb{Z}: F(S_m((x_j)_{j \in \mathbb{Z}})) = S_m(F((x_j)_{j \in \mathbb{Z}})) . \quad (4.1.1.6)$$

Definition 4.1.1.7. Linear channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **linear**, if F is a linear mapping:

$$F(\alpha(x_j)_{j \in \mathbb{Z}} + \beta(y_j)_{j \in \mathbb{Z}}) = \alpha F((x_j)_{j \in \mathbb{Z}}) + \beta F((y_j)_{j \in \mathbb{Z}}) \quad (4.1.1.8)$$

for all sequences $(x_j)_{j \in \mathbb{Z}}, (y_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and real numbers $\alpha, \beta \in \mathbb{R}$.

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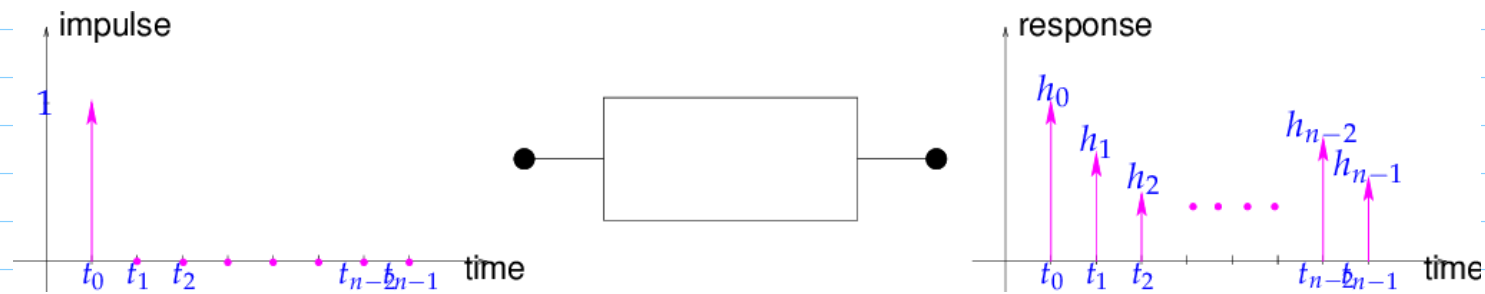
Definition 4.1.1.9. Causal channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **causal** (or physical, or nonanticipative), if the output does not start before the input

$$\forall M \in \mathbb{N}: (x_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}), x_j = 0 \quad \forall j \leq M \Rightarrow F((x_j)_{j \in \mathbb{Z}})_k = 0 \quad \forall k \leq M. \quad (4.1.1.10)$$

Definition 4.1.1.12. Impulse response

The **impulse response** (IR) of a channel/filter is the output for a single unit impulse at $t = 0$ as input, that is, the input signal is $x_j = \delta_{j,0} := \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else} \end{cases}$ (Kronecker symbol).



$$F((\delta_{j,0})_{j \in \mathbb{Z}}) = (\dots, 0, h_0, h_1, \dots, h_{n-1}, 0, \dots).$$

$$\text{T.I. : } F \circ S_m = S_m \circ F \quad \forall m \in \mathbb{Z}$$

$$F((\delta_{j,k})_{j \in \mathbb{Z}}) = (h_{j-k})_{j \in \mathbb{Z}} = \left(\dots 0 \quad \underset{\substack{\uparrow \\ t_k = k\Delta t}}{h_0} \quad h_1 \quad \dots \quad \underset{\substack{\uparrow \\ t = (k+n-1)\Delta t}}{h_{n-1}} \quad 0 \dots \right).$$

time-shifted unit pulse

For finite signal (x_j) :

$$(x_j)_{j \in \mathbb{Z}} = \sum_{k=0}^{m-1} x_k (\delta_{j,k})_{j \in \mathbb{Z}} = \sum_{k=0}^{m-1} x_k S_k((\delta_{j,0})_{j \in \mathbb{Z}}), \quad \stackrel{\text{linear combination of pulses}}{=} \quad (4.1.2.1)$$

$$F((x_j)) \stackrel{\text{linearity}}{=} \sum_k x_k F(S_k((\delta_{j,0})_{j \in \mathbb{Z}})) \stackrel{\text{T.I.}}{=} \sum_k x_k S_k \underbrace{F((\delta_{j,0})_j)}_{\text{I.R.}}$$

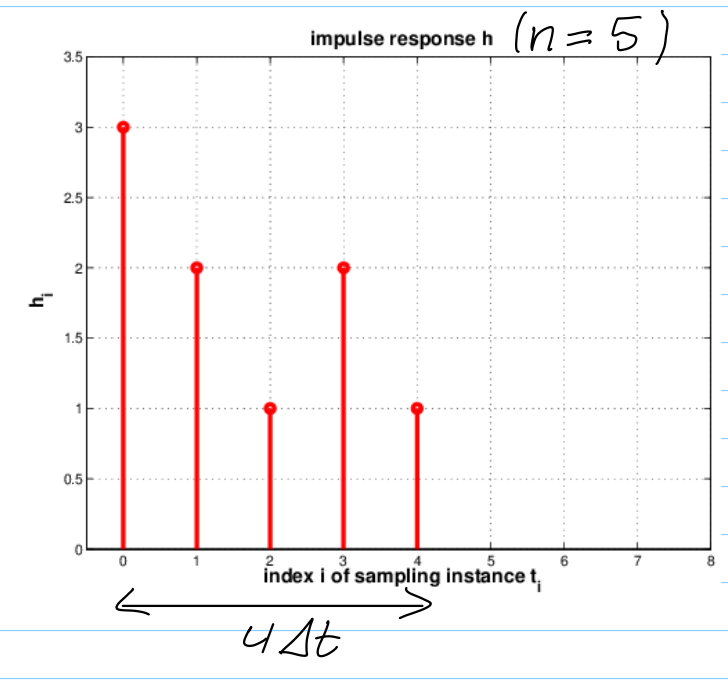
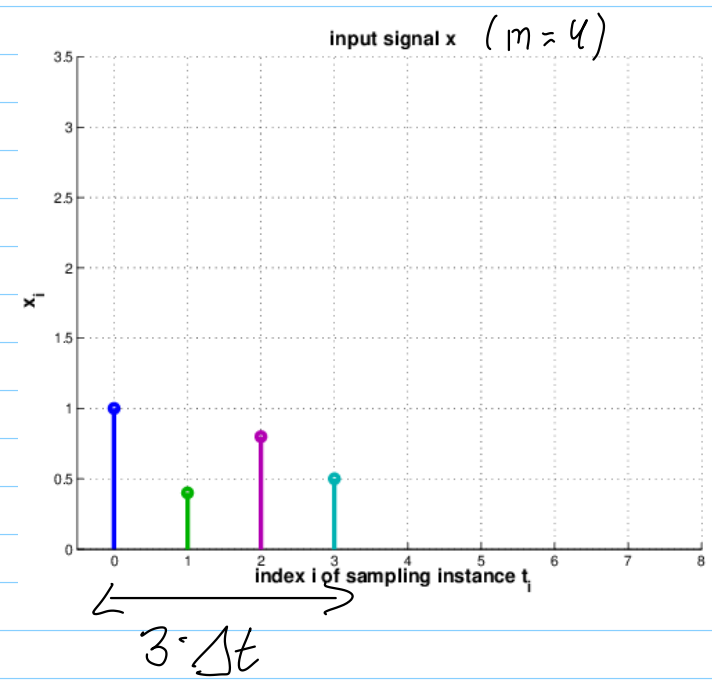
Superposition of shifted IR:

$$\begin{bmatrix} \vdots \\ 0 \\ y_0 \\ y_1 \\ \vdots \\ y_n \\ \vdots \\ y_{m+n-3} \\ y_{m+n-2} \\ 0 \\ \vdots \end{bmatrix} = x_0 \begin{bmatrix} \vdots \\ 0 \\ h_0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \end{bmatrix} + x_1 \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ h_{n-1} \\ \vdots \\ h_{n-1} \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ h_{n-1} \\ \vdots \\ 0 \\ \vdots \end{bmatrix} + \dots + x_{m-1} \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h_0 \\ \vdots \\ h_{n-1} \\ 0 \\ \vdots \end{bmatrix}. \quad (4.1.2.3)$$

$$y_k = \sum_{j=0}^{m-1} h_{k-j} x_j, \quad k = 0, \dots, m+n-2$$

[Recall : $h_k = 0$ for $k < 0$ & $k \geq n$]

③ Example:



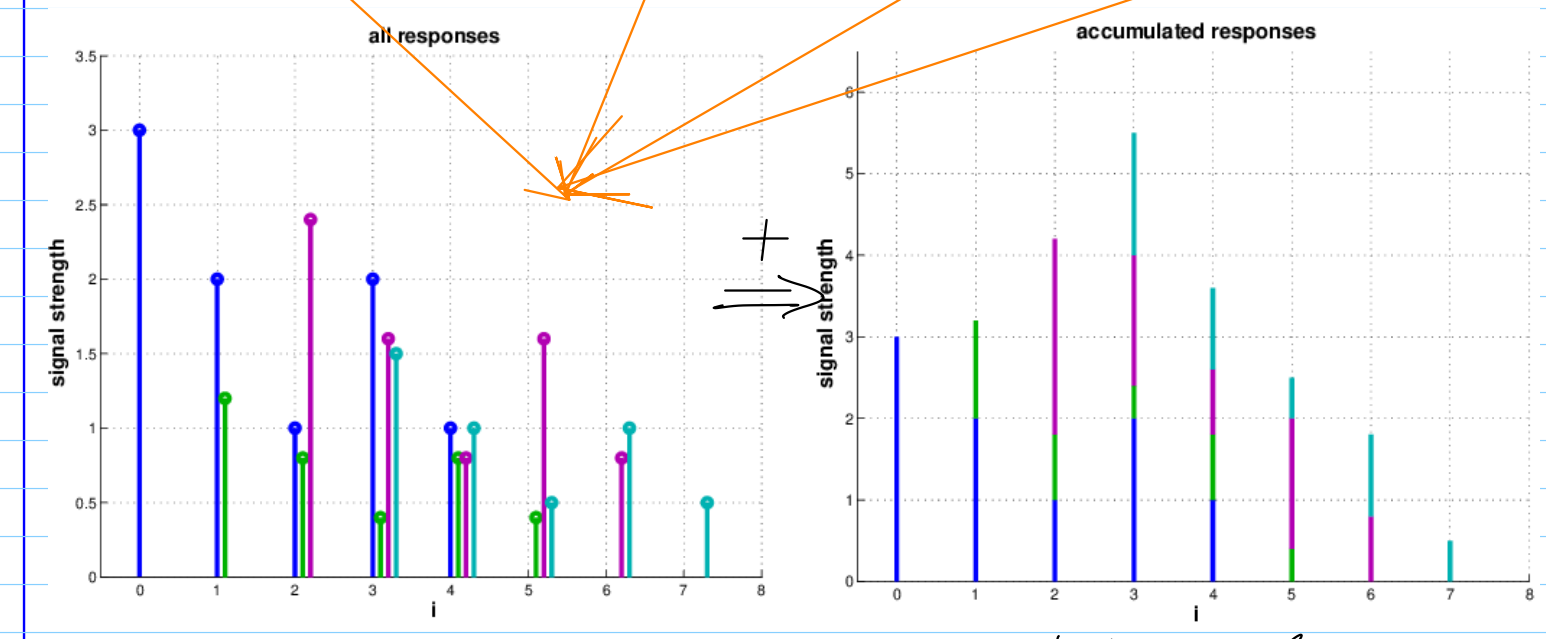
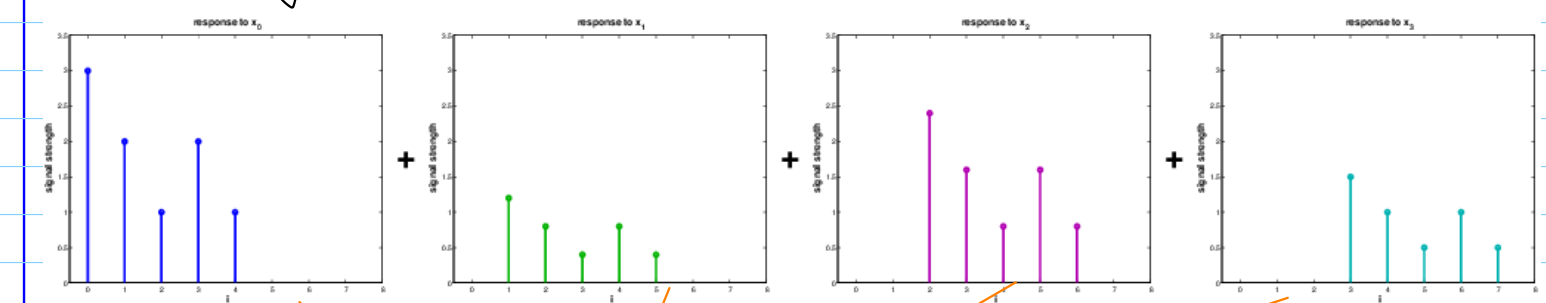
(4.1.2.3) $\hat{=}$

at $t_{m+n-2} \rightarrow$

$$\begin{bmatrix} \vdots \\ 0 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ 0 \\ \vdots \end{bmatrix} = x_0 \begin{bmatrix} \vdots \\ 0 \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + x_1 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + x_3 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ 0 \\ \vdots \end{bmatrix}$$

0 = h_5

x_k · weighted, Δt · k-time shifted IR :



Duration (output) = duration (input) + duration (IR)

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Extension to infinite signals:

$$y_k = \sum_{j=0}^{m-1} h_{k-j} x_j = \sum_{j \in \mathbb{Z}} h_{k-j} x_j$$

Definition 4.1.2.7. Convolution of sequences

Given two sequences $(h_k)_{k \in \mathbb{Z}}$, $(x_k)_{k \in \mathbb{Z}}$, at least one of which is finite or decays sufficiently fast, their **convolution** is another sequence $(y_k)_{k \in \mathbb{Z}}$, defined as

$$y_k = \sum_{j \in \mathbb{Z}} h_j x_{k-j} \quad k \in \mathbb{Z}. \quad (4.1.2.8)$$

Notation $(y_k) = (h_k) * (x_k)$

Theorem 4.1.2.9. Convolution of sequences commutes

If well-defined, the convolution of sequences commutes

$$(x_k) * (h_k) = (h_k) * (x_k).$$

Review questions 4.1.2.10

▷ Tackle them without relying on notes or the lecture document.

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