

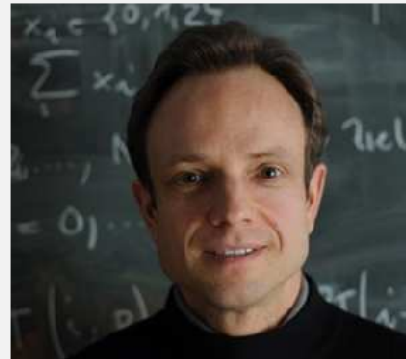
Course Video

Section 4.1.1: Discrete Finite Linear Time-Invariant Causal Channels/Filters

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

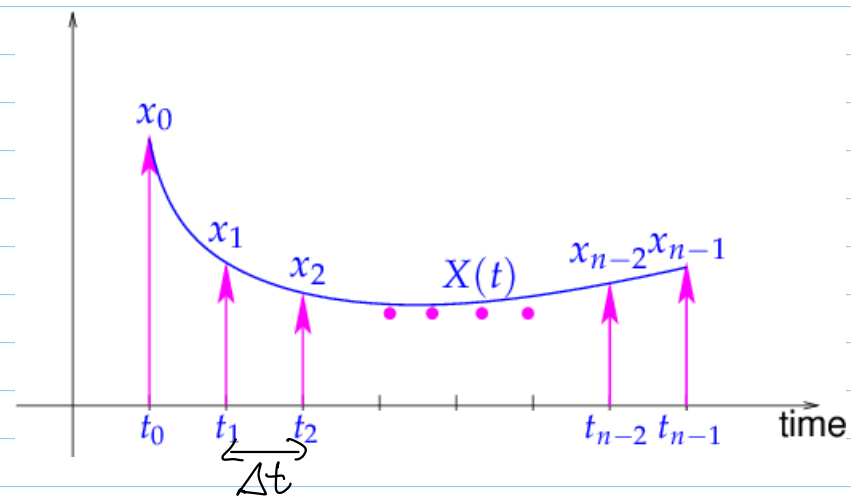
- Matrix-vector calculus from linear algebra

Duration: 20 minutes

IV. Filtering Algorithms

4.1. Filters and Convolutions

→ Numerical signal processing : LA perspective

Focus: Time-discrete, *finite*, analog signals→ vector $\in \mathbb{R}^n$ by *sampling* at
equidistant points
in time

Time-discrete,

analog signals $\leftrightarrow$ *sequence* $\in \mathbb{R}^{\mathbb{Z}}$ Notation: $(x_k)_{k \in \mathbb{Z}}$

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4.1.1. Discrete Finite Linear Time-Invariant Causal Channels/Filters

Filter/channel $\hat{=}$ $F : \{\text{input signals}\} \rightarrow \{\text{output signals}\}$

$$F : \ell^\infty(\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z})$$

$\ell^\infty(\mathbb{Z}) \hat{=}$ vector space of bounded sequences $\in \mathbb{R}^{\mathbb{Z}}$
 $\hookrightarrow \infty$ -dimensional

Definition 4.1.1.2. Finite channel/filter

A channel/filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **finite**, if every input signal of finite duration produces an output signal of finite duration,

$$\{ \exists M \in \mathbb{N} : |j| > M \Rightarrow x_j = 0 \} \Rightarrow \exists N \in \mathbb{N} : |k| > N \Rightarrow (F((x_j)_{j \in \mathbb{Z}}))_k = 0, \quad (4.1.1.3)$$

It should not matter when input arrives

Definition 4.1.1.5. Time-invariant channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **time-invariant (TI)**, if shifting the input in time leads to the same output shifted in time by the same amount; it *commutes with the time shift operator* from (4.1.1.4):

$$\forall (x_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \forall m \in \mathbb{Z} : F(S_m((x_j)_{j \in \mathbb{Z}})) = S_m(F((x_j)_{j \in \mathbb{Z}})). \quad (4.1.1.6)$$

$$S_m \hat{=}$$
 time-shift : $S_m((x_j)) = (x_{j-m})_{j \in \mathbb{Z}}$

In math :

A commuting diagram :

$$\begin{array}{ccc} \ell^\infty(\mathbb{Z}) & \xrightarrow{F} & \ell^\infty(\mathbb{Z}) \\ \downarrow S_{n1} & & \downarrow S_m \\ \ell^\infty(\mathbb{Z}) & \xrightarrow{F} & \ell^\infty(\mathbb{Z}) \end{array}$$

$\ell^\infty(\mathbb{Z})$ = a vector space $\triangleright$ linear mapping defined

Definition 4.1.1.7. Linear channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **linear**, if F is a linear mapping:

$$F(\alpha(x_j)_{j \in \mathbb{Z}} + \beta(y_j)_{j \in \mathbb{Z}}) = \alpha F((x_j)_{j \in \mathbb{Z}}) + \beta F((y_j)_{j \in \mathbb{Z}}) \quad (4.1.1.8)$$

for all sequences $(x_j)_{j \in \mathbb{Z}}, (y_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and real numbers $\alpha, \beta \in \mathbb{R}$.

Output only after onset of input

Definition 4.1.1.9. Causal channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ is called **causal** (or physical, or nonanticipative), if the output does not start before the input

$$\forall M \in \mathbb{N} : (x_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}), x_j = 0 \quad \forall j \leq M \Rightarrow F((x_j)_{j \in \mathbb{Z}})_k = 0 \quad \forall k \leq M. \quad (4.1.1.10)$$

Acronym : **LT-FIR** $\hat{=}$ linear, time-invariant, finite (& causal) filters

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Remember from LA : linear mapping uniquely defined
by action on basis vector

For $\ell^\infty(\mathbb{Z})$: basis $e_k := (\delta_{jk})_{j \in \mathbb{Z}}$: impuls

Note : $S_k(e_k) = e_0 \xrightarrow{\text{TI}} \text{knowing } F(e_0) \text{ defines}$
LT-FIR

$F(e_0) \triangleq$ impulse response (IR)

$$F((\delta_{j,0})_{j \in \mathbb{Z}}) = (\dots, 0, h_0, h_1, \dots, h_{n-1}, 0, \dots)$$

Review questions 4.1.1.13

Please answer them with consulting any notes.

4

5

7