

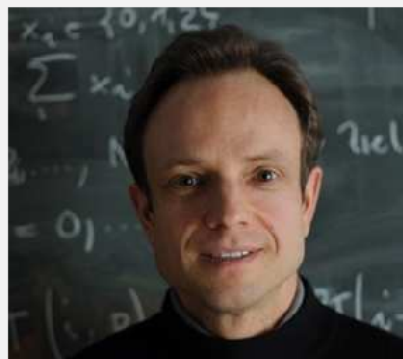
Course Video

Section 4.1.4: Periodic Convolutions

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependencies. Section 4.1.2, ??

Duration: 24 minutes

IV. Filtering Algorithms

4.1. Filters and Convolutions

4.1.4. Periodic Convolutions

Definition 4.1.2.7. Convolution of sequences

Given two sequences $(h_k)_{k \in \mathbb{Z}}, (x_k)_{k \in \mathbb{Z}}$, at least one of which is finite or decays sufficiently fast, their **convolution** is another sequence $(y_k)_{k \in \mathbb{Z}}$, defined as

$$y_k = \sum_{j \in \mathbb{Z}} h_{k-j} x_j, \quad k \in \mathbb{Z}. \quad : (y_k) = (h_k) * (x_k) \quad (4.1.2.8)$$

Definition 4.1.3.3. Discrete convolution

Given $\mathbf{x} = [x_0, \dots, x_{m-1}]^T \in \mathbb{K}^m$, $\mathbf{h} = [h_0, \dots, h_{n-1}]^T \in \mathbb{K}^n$ their **discrete convolution** is the vector $\mathbf{y} \in \mathbb{K}^{m+n-1}$ with components

$$y_k = \sum_{j=0}^{m-1} h_{k-j} x_j, \quad k = 0, \dots, 2n-2, \quad (4.1.3.4)$$

where we have adopted the convention $h_j := 0$ for $j < 0$ or $j \geq n$.

② Now: infinite signal, but periodic

Definition 4.1.4.1. Periodic time-discrete signal

An **n -periodic** signal, $n \in \mathbb{N}$, is a sequence $(x_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ satisfying

$$x_{j+n} = x_j \quad \forall j \in \mathbb{Z}.$$

$$\rightarrow \text{vector } \underline{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} \in \mathbb{R}^n$$

Action LT-FIR filter F with IR $(h_k)_{k \in \mathbb{Z}}$ on n -periodic sequence $(x_k) : (y_k) := F((x_k))$

$$y_k = \sum_{j \in \mathbb{Z}} h_{k-j} x_j = \sum_{j \in \mathbb{Z}} h_j x_{k-j}$$

$$\stackrel{j = v + \ell n}{=} \sum_{v=0}^{n-1} \sum_{\ell \in \mathbb{Z}} h_{v+\ell n} x_{k-v-\ell n} \rightarrow \text{by periodicity}$$

$$= \sum_{v=0}^{n-1} \left(\sum_{\ell \in \mathbb{Z}} h_{v+\ell n} \right) x_{k-v}$$

$$= \sum_{v=0}^{n-1} p_v x_{k-v} = (p_k) *_{\mathbb{Z}} (x_k)$$

$p_v = \sum_{\ell \in \mathbb{Z}} h_{v+\ell n} \rightarrow n\text{-periodic seq. } (p_v)_{v \in \mathbb{Z}}$

(y_k) is also n -periodic

Definition 4.1.4.7. Discrete periodic convolution

The **discrete periodic convolution** of two n -periodic sequences $(p_k)_{k \in \mathbb{Z}}, (x_k)_{k \in \mathbb{Z}}$ yields the n -periodic sequence

$$(y_k) := (p_k) *_{\mathbb{Z}} (x_k), \quad y_k := \sum_{j=0}^{n-1} p_{k-j} x_j = \sum_{j=0}^{n-1} x_{k-j} p_j, \quad k \in \mathbb{Z}.$$

$$\Leftrightarrow \underline{y} = \underline{p} *_{\mathbb{Z}} \underline{x}, \quad \underline{y}, \underline{p}, \underline{x} \in \mathbb{R}^n$$

§ 4.7.4.11 : Circulant matrices

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} p_0 & p_{n-1} & p_{n-2} & \cdots & \cdots & p_1 \\ p_1 & p_0 & p_{n-1} & \cdots & \cdots & \vdots \\ p_2 & p_1 & p_0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & \cdots & \cdots & \cdots & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} \quad (4.1.4.6)$$

$\hookrightarrow$ cyclic permutation

Definition 4.1.4.12. Circulant matrix $\rightarrow$

A matrix $\mathbf{C} = [c_{ij}]_{i,j=1}^n \in \mathbb{K}^{n,n}$ is **circulant**

$$\Leftrightarrow \exists (p_k)_{k \in \mathbb{Z}} \text{ } n\text{-periodic sequence: } c_{ij} = p_{j-i}, \quad 1 \leq i, j \leq n.$$

→ C.M. completely defined by $p \in \mathbb{K}^n$
↳ "data sparse"

4.1.4.15 Convolution by periodic convolution

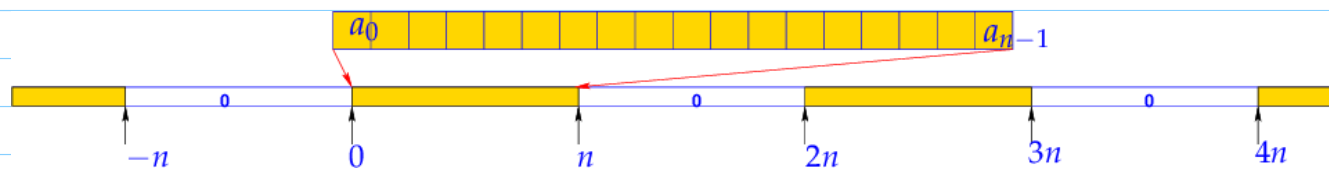
$$\underline{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix}, \quad n \in \mathbb{N}$$

$$\underline{z} = \underline{a} * \underline{b} : z_k = \sum_{j=0}^{n-1} a_{j-k} b_j, \quad k=0, \dots, 2n-2$$

Idea: Suppress interaction of different periods
by zero padding

$$x_k := \begin{cases} a_k & , \text{ if } 0 \leq k < n , \\ 0 & , \text{ if } n \leq k < 2n - 1 \end{cases} \quad , \quad y_k := \begin{cases} b_k & , \text{ if } 0 \leq k < n , \\ 0 & , \text{ if } n \leq k < 2n - 1 , \end{cases} \quad (4.1.4.16)$$

→ associated $2n-1$ -periodic sequence



$$\underline{z} = \underline{a} * \underline{b} = \underline{x} *_{2n-1} \underline{y}$$

$$\begin{bmatrix} z_0 \\ \vdots \\ \vdots \\ z_{2n-2} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & \cdots & 0 & b_{n-1} \\ b_1 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & 0 \\ b_{n-1} & 0 & \cdots & 0 & b_0 \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_{n-1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad (4.1.4.18)$$

Review questions 4.1.4.19

▷ Please answer without aids

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