

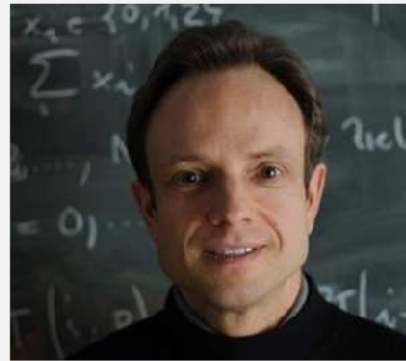
Course Video

Section 4.2.1: Diagonalizing Circulant Matrices

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- From linear algebra: eigenvalues and eigenvectors
- Computing with complex numbers

Dependencies. Section 4.1.4

Duration: 30 minutes

IV. Filtering Algorithms

4.2. Discrete Fourier Transform (DFT)

4.2.1. Diagonalizing Circulant Matrices

Recall: discrete conv. $\longleftrightarrow$ discrete periodic conv.

= multiplication with circulant matrix

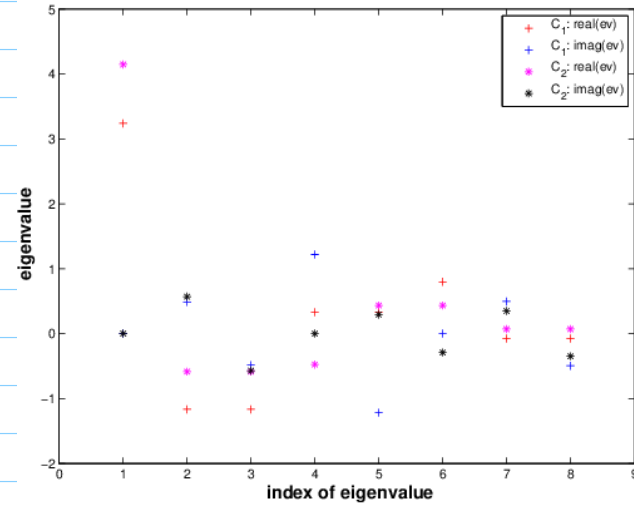
Definition 4.1.4.12. Circulant matrix $\rightarrow$ A matrix $\mathbf{C} = [c_{ij}]_{i,j=1}^n \in \mathbb{K}^{n,n}$ is **circulant** $\Leftrightarrow \exists (p_k)_{k \in \mathbb{Z}}$ n -periodic sequence: $c_{ij} = p_{j-i}, 1 \leq i, j \leq n$.

$$\text{circul}(\mathbf{p}) = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & \cdots & p_{n-1} \\ p_{n-1} & p_0 & & & & p_{n-2} \\ p_{n-2} & & & & & \vdots \\ \vdots & & & & & \\ \vdots & & & & & \\ p_2 & & & & & p_1 \\ p_1 & p_2 & \cdots & \cdots & p_{n-1} & p_0 \end{bmatrix}$$

②

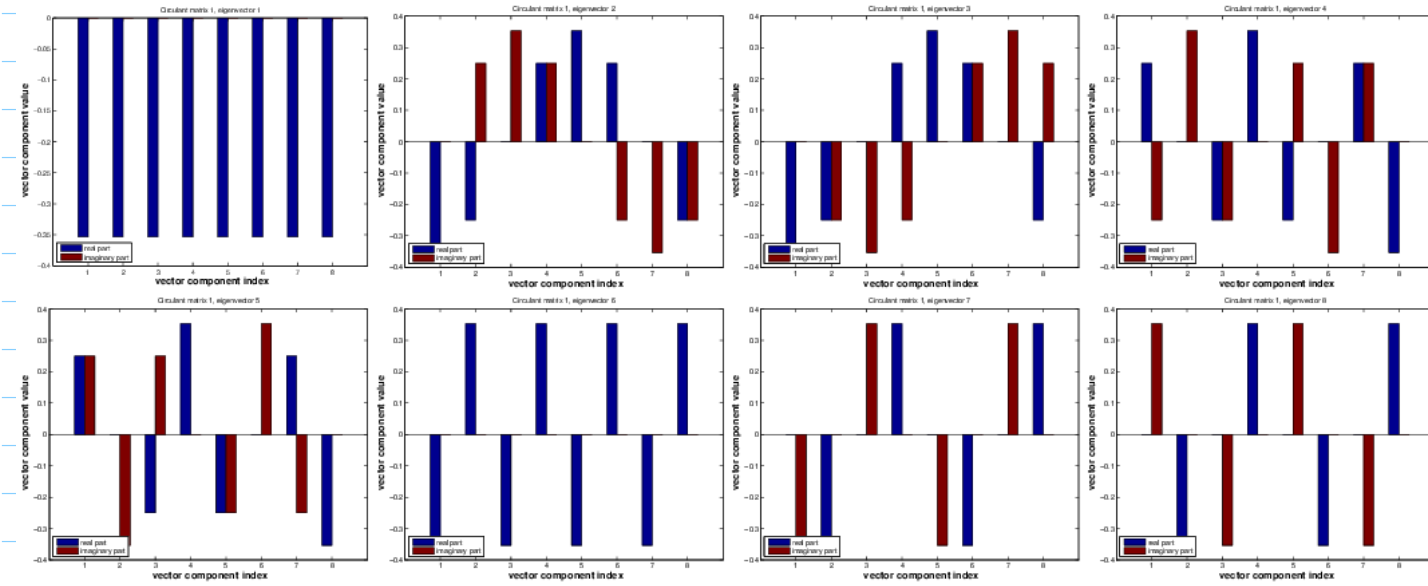
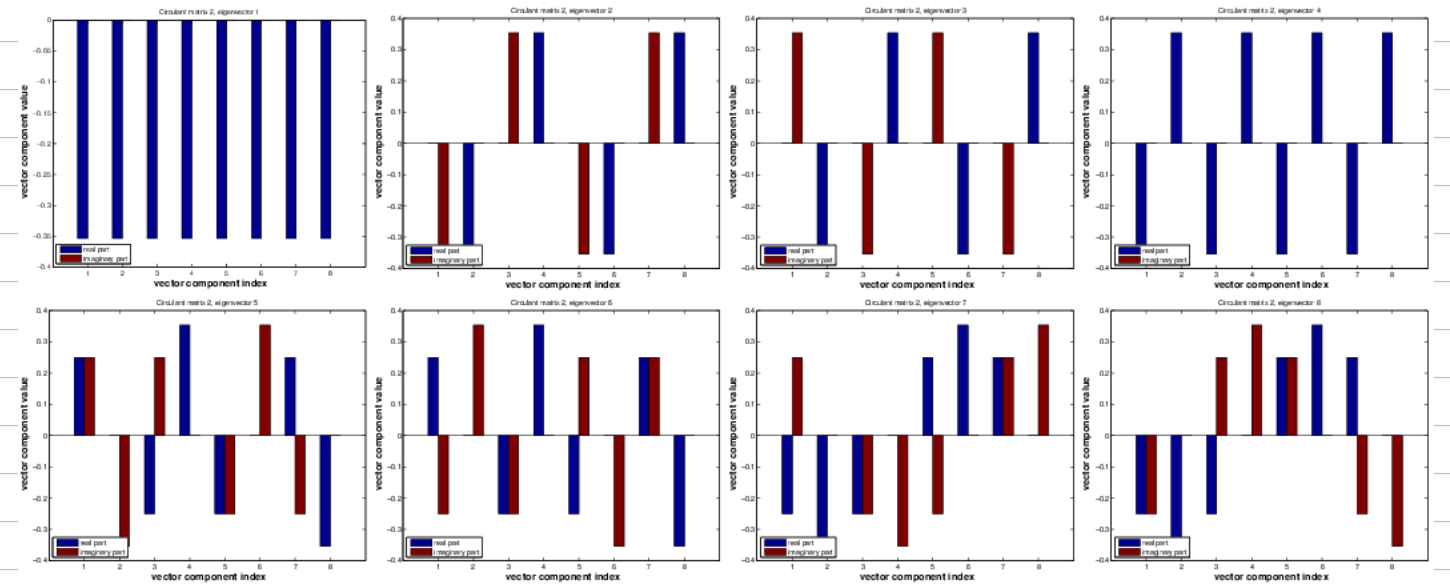
Exp.: 4.2.0.1 Eigenvectors (EVs) of circulant matrices

$$C = \text{circul}(\text{VectorXd}::\text{Random}(8))$$



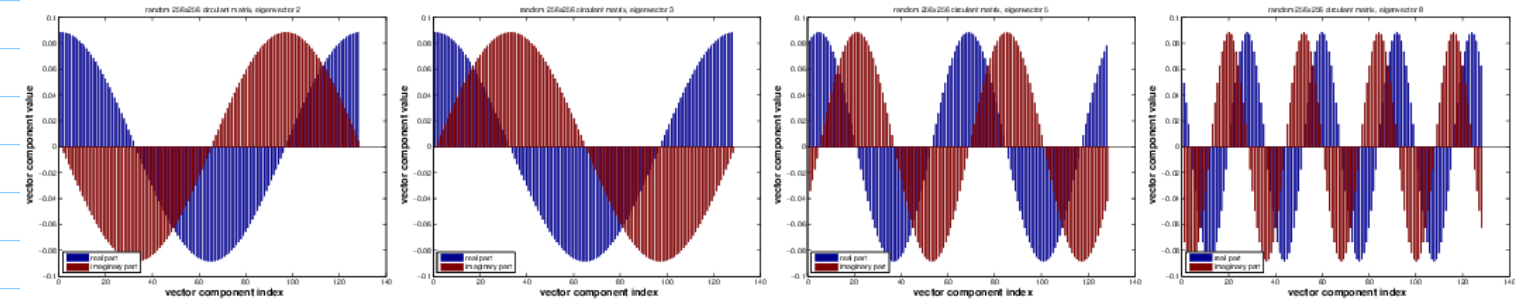
△ different (complex) eigenvalues

Eigenvectors of matrix C_1 , visualized through the size of the real and imaginary parts of their components.


 C_2 :


▷ All $n \times n$ circulant matrices have the same eigenvectors $\in \mathbb{C}^n$

Eigenvectors of circulant matrix $C = \text{circul}([1, 2, \dots, 128]^T)$:



▷ Reminds of sin/cos

③ § 4.2.0.6 EV circulant matrices

Recall: $e^{it} = \cos t + i \sin t$

n -th root of unity $\omega_n = e^{-\frac{2\pi i}{n}}$, $n \in \mathbb{N}$

satisfies $\bar{\omega}_n = \omega_n^{-1}$, $\omega_n^n = 1$, $\omega_n^{n/2} = -1$, $\omega_n^k = \omega_n^{k+n} \forall k \in \mathbb{Z}$, (4.2.1.7)

III $\sum_{k=0}^{n-1} \omega_n^{kj} = \sum_{k=0}^{n-1} (\omega_n^j)^k = \begin{cases} n & \text{if } j \equiv 0 \pmod{n} \\ 0 & \text{if } j \not\equiv 0 \pmod{n} \end{cases}$ by geometric sum formula (4.2.1.8)

$\omega_n^{kk} = (\omega_n^k)^k$

$C \in \mathbb{C}^{n,n}$: $(C)_{i,j} = u_{i-j}$, $(u_k) \hat{=} n$ -periodic

$\underline{v}_k := [\omega_n^{-kj}]_{j=0}^{n-1} \in \mathbb{C}^n$, $k = 0, \dots, n-1$

$$\begin{aligned} (C \underline{v}_k)_j &= \sum_{\ell=0}^{n-1} (C)_{j,\ell} (\underline{v}_k)_\ell \\ &= \sum_{\ell=0}^{n-1} u_{j-\ell} \omega_n^{-k\ell} \\ \ell \leftarrow j-\ell &= \sum_{\ell=0}^{n-1} u_\ell \omega_n^{(j-\ell)k} \\ &= \omega_n^{-jk} \cdot \underbrace{\sum_{\ell=0}^{n-1} u_\ell \omega_n^{\ell k}}_{= \lambda_k} = \lambda_k (\underline{v}_k)_j \end{aligned}$$

$C \underline{v}_k = \lambda_k \underline{v}_k$: EV!

Set of EVs of C :

$$\{\underline{v}_0, \dots, \underline{v}_{n-1}\} = \left\{ \begin{bmatrix} \omega_n^0 \\ \vdots \\ \omega_n^0 \end{bmatrix}, \begin{bmatrix} \omega_n^0 \\ \omega_n^{-1} \\ \vdots \\ \omega_n^{1-n} \end{bmatrix}, \dots, \begin{bmatrix} \omega_n^0 \\ \omega_n^{2-n} \\ \omega_n^{2(2-n)} \\ \vdots \\ \omega_n^{-(n-1)(n-2)} \end{bmatrix}, \begin{bmatrix} \omega_n^0 \\ \omega_n^{1-n} \\ \omega_n^{2(1-n)} \\ \vdots \\ \omega_n^{-(n-1)^2} \end{bmatrix} \right\}. \quad (4.2.1.11)$$

orthogonal

$$\underline{v}_k^H \underline{v}_m = \sum_{j=0}^{n-1} \omega_n^{kj} \omega_n^{-mj} = \sum_{j=0}^{n-1} \omega_n^{(k-m)j} = 0, \text{ if } k \neq m$$

$\rightarrow$ Fourier matrix $\rightarrow$ diagonalizes circulant matrices

$$\mathbf{F}_n = \begin{bmatrix} \omega_n^0 & \omega_n^0 & \dots & \omega_n^0 \\ \omega_n^0 & \omega_n^1 & \dots & \omega_n^{n-1} \\ \omega_n^0 & \omega_n^2 & \dots & \omega_n^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^0 & \omega_n^{n-1} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} = [\omega_n^{lj}]_{l,j=0}^{n-1} \in \mathbb{C}^{n,n}. \quad (4.2.1.13)$$

Lemma 4.2.1.14. Properties of Fourier matrix

The scaled Fourier-matrix $\frac{1}{\sqrt{n}} \mathbf{F}_n$ is unitary ($\rightarrow$ Def. 6.2.1.2): $\mathbf{F}_n^{-1} = \frac{1}{n} \mathbf{F}_n^H = \frac{1}{n} \bar{\mathbf{F}}_n$.

$$\begin{aligned} C \underline{v}_k &= C (\bar{\mathbf{F}}_n)_{:,k} = \lambda_k (\bar{\mathbf{F}}_n)_{:,k} \\ C \bar{\mathbf{F}}_n &= \bar{\mathbf{F}}_n \mathbf{D}, \quad \mathbf{D} = \text{diag}(\mathbf{F}_n \underline{u}) \end{aligned}$$

Lemma 4.2.1.16. Diagonalization of circulant matrices ($\rightarrow$ Def. 4.1.4.12)

For any circulant matrix $\mathbf{C} \in \mathbb{K}^{n,n}$, $c_{ij} = u_{i-j}$, $(u_k)_{k \in \mathbb{Z}}$ n -periodic sequence, holds true

$$\mathbf{C}\bar{\mathbf{F}}_n = \bar{\mathbf{F}}_n \text{diag}(d_1, \dots, d_n) \quad , \quad [d_0, \dots, d_{n-1}]^\top = \mathbf{F}_n[u_0, \dots, u_{n-1}]^\top .$$

$\triangleright \quad \mathbf{C} = \mathbf{F}_n^{-1} \mathbf{D} \mathbf{F}_n$

$\rightarrow$ All operations w/ circulant matrices can be realized by means of Fourier matrices

Definition 4.2.1.18. Discrete Fourier transform (DFT)

The linear map $\text{DFT}_n : \mathbb{C}^n \mapsto \mathbb{C}^n$, $\text{DFT}_n(\mathbf{y}) := \mathbf{F}_n \mathbf{y}$, $\mathbf{y} \in \mathbb{C}^n$, is called **discrete Fourier transform** (DFT), i.e. for $[c_0, \dots, c_{n-1}] := \text{DFT}_n(\mathbf{y})$

$$c_k = \sum_{j=0}^{n-1} y_j \omega_n^{kj} \quad , \quad k = 0, \dots, n-1. \quad (4.2.1.19)$$

$$\mathbf{F}_n^{-1} = \frac{1}{n} \bar{\mathbf{F}}_n \Rightarrow \leq = \text{DFT}(\mathbf{y}) \Leftrightarrow y_k = \frac{1}{n} \sum_{j=0}^{n-1} c_j \omega_n^{-jk}$$

Another "DFT"!

§ 4.2.1.21 DFT in Eigen

#include <unsupported/Eigen/FFT>

C++ code 4.2.1.22: Demo: discrete Fourier transform in EIGEN $\rightarrow$ GITLAB

```

2 int main() {
3     using Comp = complex<double>;
4     const VectorXcd::Index n = 5;
5     VectorXcd y(n), c(n), x(n);
6     y << Comp(1,0), Comp(2,1), Comp(3,2), Comp(4,3), Comp(5,4);
7     FFT<double> fft; // DFT transform object
8     c = fft.fwd(y); // DTF of y, see Def. 4.2.1.18
9     x = fft.inv(c); // inverse DFT of c, see (4.2.1.20)
10
11     cout << "y = " << y.transpose() << endl
12           << "c = " << c.transpose() << endl
13           << "x = " << x.transpose() << endl;
14     return 0;
15 }
```

Review questions 4.2.1.23
(Answer without aids)

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