

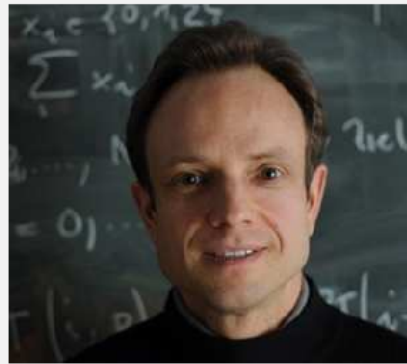
Course Video

Section 4.5: Toeplitz Matrix Techniques

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

Dependencies. Section 3.1.2, Section 4.1.4, Section 4.2.1

Duration: 35 minutes

IV. Filtering Algorithms

4.5. Toeplitz Matrix Techniques

4.5.1. Matrices with Constant Diagonals

Definition 4.5.1.7. Toeplitz matrix

$\mathbf{T} = (t_{ij})_{i,j=1}^n \in \mathbb{K}^{m,n}$ is a **Toeplitz matrix**, if there is a vector $\mathbf{u} = [u_{-m+1}, \dots, u_{n-1}] \in \mathbb{K}^{m+n-1}$ such that $t_{ij} = u_{j-i}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

$$\mathbf{T} = \begin{bmatrix} u_0 & u_1 & \dots & \dots & u_{n-1} \\ u_{-1} & u_0 & u_1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ u_{1-m} & \dots & \dots & u_{-1} & u_0 \end{bmatrix}$$

▷ $m+n-1$ numbers to describe an $m \times n$ Toeplitz matrix: *data-sparse*

②

Recall : LT-FIR channels
(linear, time-invariant, finite, causal)

If $(h_0, \dots, h_{n-1}) = \text{impulse response}$

▷ matrix model : $(m > n)$

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{m+n-2} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & & 0 \\ h_1 & & & \\ & \ddots & & \\ h_{n-1} & & & 0 \\ 0 & h_0 & & \\ & h_1 & & \\ & & \ddots & \\ 0 & & & h_{n-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{m-1} \end{bmatrix} =: \mathbf{C} \mathbf{x}.$$

↑
Toeplitz matrix

↑
input signal

↑
output signal

(4.1.3.2)

Ex. 4.5.1.1 : Parameter identification for LT-FIR channels

Given : • m -periodic input signal (x_k)
• m -periodic output signal (y_k)
(both measured $\Rightarrow$ affected by measurement error)

$$m > n$$

cf. (4.1.2.4)

$$\exists \mathbf{h} = [h_0, \dots, h_{n-1}]^T \in \mathbb{R}^n, \quad n \leq m : y_k = \sum_{j=0}^{n-1} h_j x_{k-j}, \quad k=0, \dots, m-1 \quad (4.5.1.2)$$

↑
impulse response : to be estimated

↳ overdetermined LSE : least squares estimator

$$\| \mathbf{A} \mathbf{h} - \mathbf{y} \|_2 = \left\| \begin{bmatrix} x_0 & x_{-1} & \dots & \dots & x_{1-n} \\ x_1 & x_0 & x_{-1} & & \vdots \\ \vdots & x_1 & x_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_{-1} \\ x_{n-1} & x_n & x_{n-1} & \dots & x_1 & x_0 \\ \vdots & & & \ddots & \vdots & \vdots \\ x_{m-1} & \dots & \dots & \dots & x_{m-n} \end{bmatrix} \begin{bmatrix} h_0 \\ \vdots \\ h_{n-1} \end{bmatrix} - \begin{bmatrix} y_0 \\ \vdots \\ y_{m-1} \end{bmatrix} \right\|_2 \rightarrow \min.$$

③ Normal equations : $A^T A \underline{h} = A^T \underline{y}$

$$(A^T A)_{i,j} = \sum_{k=0}^{m-1} x_{k-i} x_{k-j} = \sum_{k=0}^{m-1} x_k x_{k-(j-i)} = z_{i-j}$$

$\uparrow$
 (x_k) m-periodic

$\uparrow$
is a **Toeplitz matrix**

4.5.2 Toeplitz Matrix Arithmetic

Compute Tx , $x \in \mathbb{R}^n$, $T \in \mathbb{R}^{m,n}$ a Toeplitz matrix

$$T = [u_{j-i}] \in \mathbb{R}^{m,n}$$

$(u_{-m+1}, \dots, u_{n-1}) \triangleq$ defining sequence

Idea: Embedding into a **circulant matrix**

T as sub-block of $(m+n) \times (m+n)$ **circulant matrix**

For $m = n$:

$$C = \begin{bmatrix} u_0 & u_1 & \dots & \dots & u_{n-1} & 0 & u_{1-n} & \dots & \dots & u_{-1} \\ u_{-1} & u_0 & u_1 & \dots & \vdots & u_{n-1} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & u_1 & \vdots & \ddots & \ddots & \ddots & u_{1-n} \\ u_{1-n} & \dots & \dots & u_{-1} & u_0 & u_1 & \dots & \dots & u_{n-1} & 0 \\ 0 & u_{1-n} & \dots & \dots & u_{-1} & u_0 & u_1 & \dots & \dots & u_{n-1} \\ u_{n-1} & 0 & \ddots & \ddots & \vdots & u_{-1} & u_0 & u_1 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & u_{1-n} & \vdots & \ddots & \ddots & \ddots & u_1 \\ u_1 & \dots & \dots & u_{n-1} & 0 & u_{1-n} & \dots & \dots & u_{-1} & u_0 \end{bmatrix}$$

spanned by $(m+n)$ -periodic seq. $(u_0, \dots, u_{m-1}, 0, u_{n-1}, \dots, u_1)$

$$\underbrace{C}_{(m+n) \times (m+n)} \underbrace{\begin{bmatrix} x \\ n \end{bmatrix}}_{(m+n) \times 1} = \begin{bmatrix} Tx \\ * \end{bmatrix}$$

$\triangleq$ discrete periodic convolution: DFT via FFT

$\triangleright$ Cost = $O((m+n) \log(m+n))$ for $m, n \rightarrow \infty$

4.5.3. The Levinson Algorithm

$T = [u_{j-i}] \in \mathbb{R}^{n,n}$ Toeplitz, *s.p.d.*, $u_0 = 1$
Solve LSE $Tx = \underline{b} = [b_1, \dots, b_n]^T \in \mathbb{R}^n$

Idea: Recursion based on block partitioning

$T_k = [u_{j-i}]_{i,j=0}^{k-1} \in \mathbb{R}^{k,k}$, *s.p.d.* Toeplitz

$\underline{u}^k = [u_1, \dots, u_k]^T \in \mathbb{R}^k$ $1 \leq k \leq n$

$\underline{b}^k = [b_1, \dots, b_k]^T \in \mathbb{R}^k$

$\underline{x}^k \in \mathbb{R}^k$; $T_k \underline{x}^k = \underline{b}^k$

(k,1)-partitioned LSE $\uparrow$

$$T_{k+1} \underline{x}^{k+1} = \begin{bmatrix} \overbrace{[u_k \cdots u_1]}^{T_k} & \begin{bmatrix} u_k \\ \vdots \\ u_1 \\ 1 \end{bmatrix} \\ \underline{u}^{k+1} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \tilde{\underline{x}}^{k+1} \\ x_{k+1}^{k+1} \end{bmatrix}}_{\underline{x}^{k+1}} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \end{bmatrix} = \begin{bmatrix} \underline{b}^k \\ b_{k+1} \end{bmatrix} = \underline{b}^{k+1} \quad (4.5.3.1)$$

Next: block elimination

reverse() - permutation
 $\downarrow$

$$\begin{cases} \tilde{\underline{x}}_{k+1} = T_k^{-1}(\underline{b}^k - x_{k+1}^{k+1} P_k \underline{u}^k) = \underline{x}^k - x_{k+1}^{k+1} T_k^{-1} P_k \underline{u}^k, \\ x_{k+1}^{k+1} = b_{k+1} - P_k \underline{u}^k \cdot \tilde{\underline{x}}^{k+1} = b_{k+1} - (P_k \underline{u}^k)^T \underline{x}^k + x_{k+1}^{k+1} (P_k \underline{u}^k)^T T_k^{-1} P_k \underline{u}^k. \end{cases} \quad (4.5.3.3)$$

$$\underline{x}^{k+1} = \begin{bmatrix} \tilde{\underline{x}}^{k+1} \\ x_{k+1}^{k+1} \end{bmatrix} \text{ with } \begin{cases} x_{k+1}^{k+1} = (b_{k+1} - P_k \underline{u}^k) / \sigma_k \\ \tilde{\underline{x}}^{k+1} = \underline{x}^k - x_{k+1}^{k+1} \underline{y}^k \end{cases}, \quad \sigma_k := 1 - P_k \underline{u}^k \cdot \underline{y}^k. \quad (4.5.3.4)$$

$$T_k \underline{y}^k = P_k \underline{u}^k$$

$\neq 0$, if T_k s.p.d.

Two recursive calls? $\rightarrow$ Too expensive

Idea: If $\underline{b} = P_n \underline{u}^n \Rightarrow \underline{x}^k = \underline{y}^k$

$\rightarrow$ Simple linear recursion: $\underline{y}^k$ with cost $= O(k^2)$

Then compute $\underline{x}^k$: another simple linear rec.

cost $= O(k^2)$, for $k \rightarrow \infty$

$\rightarrow$ Review questions 4.5.3.8

5

7