

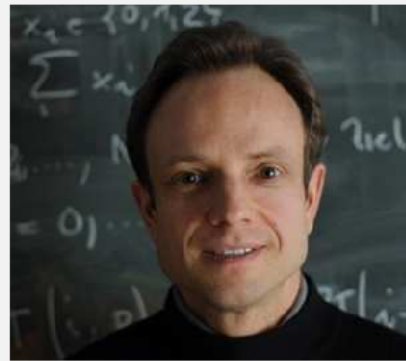
# Course Video

## Section 5.1: Abstract Interpolation

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(C) Seminar für Angewandte Mathematik, ETH Zürich



### Prerequisites.

- Linear systems of equations
- Notion of continuous functions

Duration: 26 minutes

Note: Possible minor *mismatch of video and tablet notes!*

[Corrections and updates can be incorporated into tablet notes only]



# V. Machine Learning of One - Dimensional Data

## (Data Interpolation and Data Fitting in 1D)

### 5.1. Abstract Interpolation (AI)

#### One-dimensional interpolation

Given: data points  $(t_i, y_i)$ ,  $i = 0, \dots, n$ ,  $n \in \mathbb{N}$ ,  $t_i \in I \subset \mathbb{R}$ ,  $y_i \in \mathbb{R}$

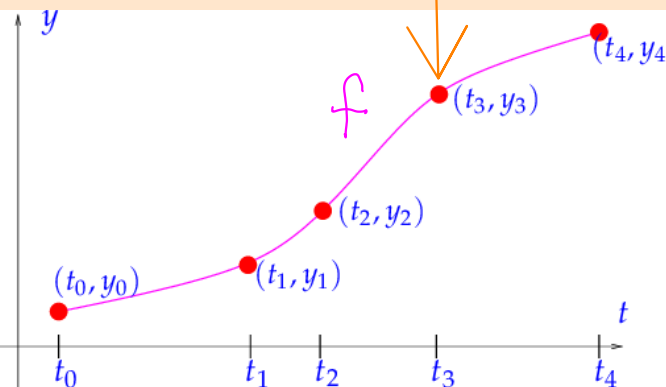
Objective: Reconstruction of a function  $f : I \mapsto \mathbb{R}$

- satisfying the  $n + 1$  **interpolation conditions** (IC)  $\rightarrow$  defines smoothness requirements etc.

$$f(t_i) = y_i, \quad i = 0, \dots, n. \quad (5.1.0.2)$$

- and belonging to a set  $V$  of eligible functions.

The function  $f$  we find is called the **interpolant** of the given data set  $\{(t_i, y_i)\}_{i=0}^n$ .



$t_i \hat{=}$  nodes

$y_i \hat{=}$  values

(2)

## Ex 5.1.0.8: Constitutive relations from measurements

Examples:  $t$  and  $y$  could be

| $t$                | $y$               |
|--------------------|-------------------|
| voltage $U$        | current $I$       |
| pressure $p$       | density $\rho$    |
| magnetic field $H$ | magnetic flux $B$ |
| ...                | ...               |

↑ accurate  
↕ IC

Functional relation  $y = f(t)$  needed for numerical

## Rem. 5.1.0.9/5.1.0.11: Functions in a code

→ OO-approach: Functor object

## C++-code 5.1.0.12: C++ class representing an interpolant in 1D

```

1 class Interpolant {
2   private:
3     // Various internal data describing f
4     // Can be the coefficients of a basis representation (5.1.0.14)
5   public:
6     // Constructor: computation of coefficients  $c_j$  of representation
7     // (5.1.0.14)
8     Interpolant(const vector<double>& t, const vector<double>& y);
9     // Evaluation operator for interpolant  $f \rightarrow f(t)$ 
10    double operator()(double t) const;
11 };

```

Issue: Efficient implementation  
build an internal representation of  $f$

Special case

$$V = \text{Span} \{ \{t \mapsto b_0(t)\}, \dots, \{t \mapsto b_m(t)\} \} = \text{Span} \{b_0, \dots, b_m\}$$

a basis  $\Rightarrow \dim V = m+1$

$$\forall f \in V \Leftrightarrow \exists \alpha_j \in \mathbb{R} : f(t) = \sum_{j=0}^n \alpha_j b_j(t) \quad (5.1.0.14)$$

$$\text{IC: } f(t_i) = y_i \quad \mapsto \text{basis expansion coefficients}$$

$$(5.1.0.2) \text{ \& } (5.1.0.14) \Rightarrow f(t_i) = \sum_{j=0}^m \alpha_j b_j(t_i) = y_i, \quad i = 0, \dots, n, \quad (5.1.0.22)$$

$$\mathbf{A} \mathbf{c} := \begin{bmatrix} b_0(t_0) & \dots & b_m(t_0) \\ \vdots & & \vdots \\ b_0(t_n) & \dots & b_m(t_n) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} =: \mathbf{y} \quad (5.1.0.23)$$

This is an  $(m+1) \times (n+1)$  linear system of equations!Unique solution  $\forall y$  only if  $m = n$ 

$$m = n, A \text{ regular} \Rightarrow f(t) = \sum_{j=0}^n (A^{-1} \mathbf{y})_j b_j(t)$$

▷ Linear interpolation operator  $\left\{ \begin{array}{l} \mathbb{R}^{n+1} \longrightarrow V \\ \mathbf{y} \longmapsto f \end{array} \right.$   
↳ as a mapping

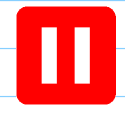
③

continuous function  $\Sigma [t_0, t_m] \rightarrow \mathbb{R}$

**Definition 5.1.0.25. Linear interpolation operator**

An interpolation operator  $I : \mathbb{R}^{n+1} \mapsto C^0([t_0, t_m])$  for the given nodes  $t_0 < t_1 < \dots < t_n$  is called linear, if

$$I(\alpha y + \beta z) = \alpha I(y) + \beta I(z) \quad \forall y, z \in \mathbb{R}^{n+1}, \alpha, \beta \in \mathbb{R}. \quad (5.1.0.26)$$



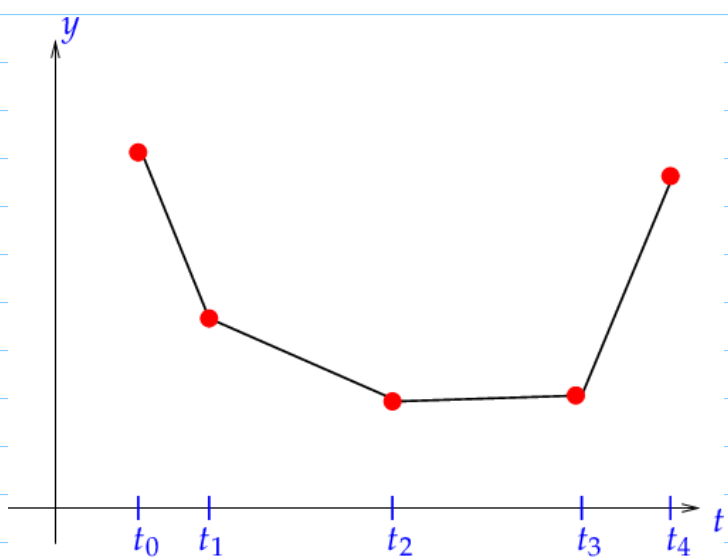
When will  $A = I$  ?

$$[A]_{ij} = b_j(t_i)$$

$b_i(t_j) = \delta_{ij}$  : a cardinal basis

$$\hookrightarrow f = \sum_{j=0}^n \gamma_j b_j$$

Ex. 5.1.0.15 : Piecewise linear interpolation

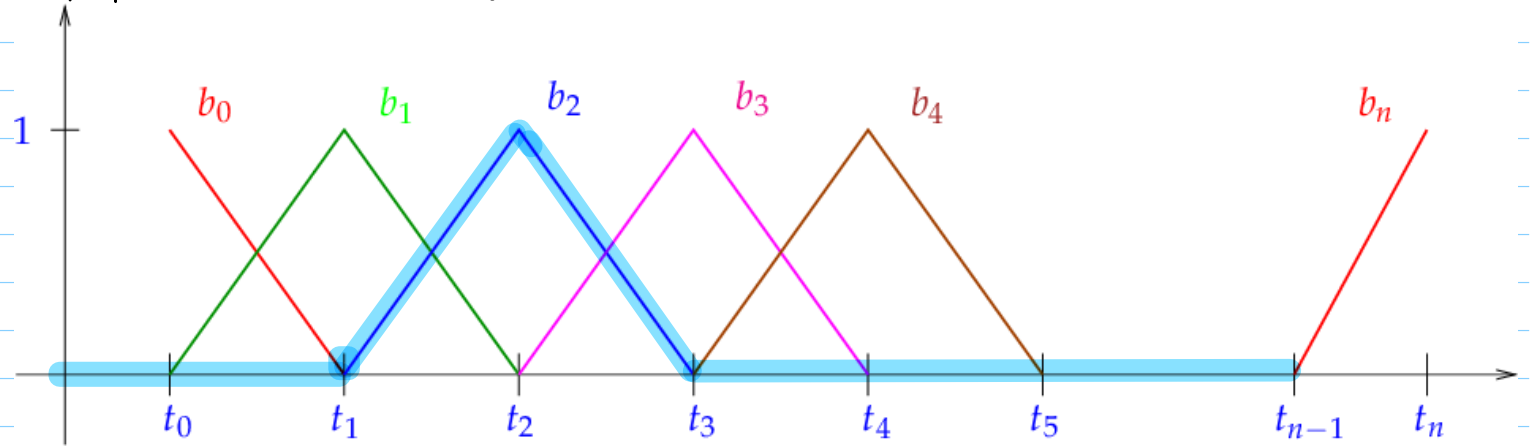


Graph of interpolant  
 $\Updownarrow$   
 polygon through data points

(5.1.0.16)

$$\begin{aligned} V &= \{ f \in C^0([t_0, t_m]) : f(t) = \underbrace{\alpha_i + \beta_i t}_{\text{on } [t_i, t_{i+1}]}, \alpha_i, \beta_i \in \mathbb{R} \} \\ \uparrow \\ \dim &= m+1 \end{aligned}$$

A cardinal basis : "tent functions"



$$b_2(t_2) = 1$$

④

▷ Review questions 5.1.0.27  
( Try to answer without consulting notes )

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