

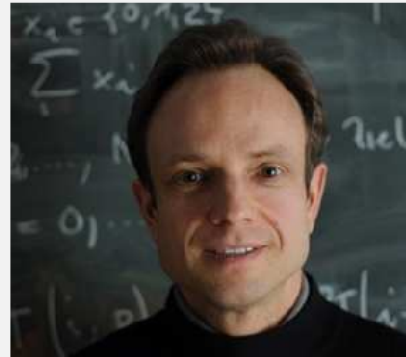
Course Video

Section 5.2.4: Polynomial Interpolation:
Sensitivity

Prof. R. Hiptmair, SAM, ETH Zurich

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Norms for functions
- Norms of linear mappings

Dependencies. Section 2.2.2, Section 5.2.2

Duration: 23 minutes

V. Data Interpolation and
Data Fitting in 1D

5.2. Global Polynomial Interpolation

5.2.4. Polynomial Interpolation : Sensitivity

Sensitivity $\leftrightarrow$ amplification of perturbations of data
(of a problem)

$\hookrightarrow$ A mapping : $\{\text{Data (space)}\} \rightarrow \{\text{Result (space)}\}$

For LIP : $\begin{cases} \text{Data points} : (\mathbb{R} \times \mathbb{R})^{n+1} \\ \text{Data values} : \mathbb{R}^{n+1} \end{cases} \longrightarrow \mathcal{P}_n$
(a function)

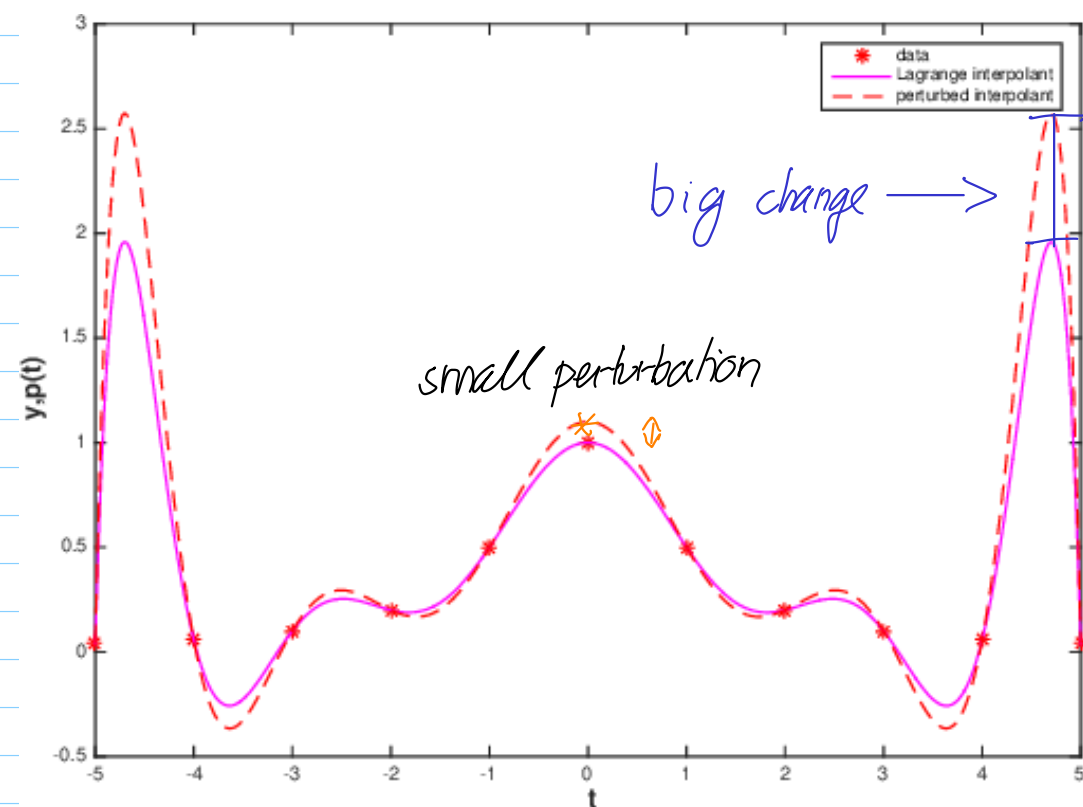
$\not\hookrightarrow \{p : I \subset \mathbb{R} \rightarrow \mathbb{R}\}$

Our focus is on **interpolation operator**

$$I_T : \begin{cases} \mathbb{R}^{n+1} & \rightarrow \mathcal{P}_n, \\ (y_0, \dots, y_n)^T & \mapsto \text{interpolating polynomial } p. \end{cases} \quad (5.2.2.9)$$

②

Ex 5.2.4.1 (Oscillating polynomial interpolant)



- possible strong oscillations of interpolating polynomials of high degree on uniformly spaced nodes!
- Slight perturbations of data values can engender strong variations of a high-degree Lagrange interpolant "far away".



Needed for quantifying perturbations **norms**

- On data space $\mathbb{R}^{n+1}$: $\|y\|_2, \|y\|_\infty$
- On result space $\mathcal{P}_n \triangleq$ a function space on $I \subset \mathbb{R}$

supremum norm $\|f\|_{L^\infty(I)} := \sup\{|f(t)| : t \in I\}, \quad (5.2.4.5)$

L^2 -norm $\|f\|_{L^2(I)}^2 := \int_I |f(t)|^2 dt, \quad \longleftrightarrow \|y\|_2 \quad (5.2.4.6)$

L^1 -norm $\|f\|_{L^1(I)} := \int_I |f(t)| dt. \quad \longleftrightarrow \|y\|_1 \quad (5.2.4.7)$

§ 5.2.4.7: Sensitivity of linear (problem) maps

$$I_\tau(y + \delta y) = I_\tau(y) + I_\tau(\delta y)$$

$\hookrightarrow$ perturbation: $\delta y \rightarrow I_\tau(\delta y)$

Sensitivity: $\sup_{\delta y \neq 0} \frac{\|I_\tau(\delta y)\|}{\|\delta y\|}$

$\triangleq$ operator norm of I_τ

③ ∞ -sensitivity

$$\|I_{\mathcal{T}}(\mathbf{y})\|_{L^\infty(I)} = \sup_{t \in I} \left| \sum_{j=0}^n y_j \overset{\text{Lagrange polynomial}}{\downarrow} L_j(t) \right|$$

$$\triangle\text{-inequ.} \leq \sum_{j=0}^n |y_j| \|L_j\|_{L^\infty(I)}$$

$$\leq \|\mathbf{y}\|_\infty \sum_{j=0}^n \|L_j\|_{L^\infty(I)}$$

Lemma 5.2.4.10. Absolute conditioning of polynomial interpolation

Given a mesh $\mathcal{T} \subset \mathbb{R}$ with generalized Lagrange polynomials $L_i, i = 0, \dots, n$, and fixed $I \subset \mathbb{R}$, the norm of the interpolation operator satisfies

$$\|I_{\mathcal{T}}\|_{\infty \rightarrow \infty} := \sup_{\mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\|I_{\mathcal{T}}(\mathbf{y})\|_{L^\infty(I)}}{\|\mathbf{y}\|_\infty} = \left\| \sum_{i=0}^n |L_i| \right\|_{L^\infty(I)}, \quad (5.2.4.11)$$

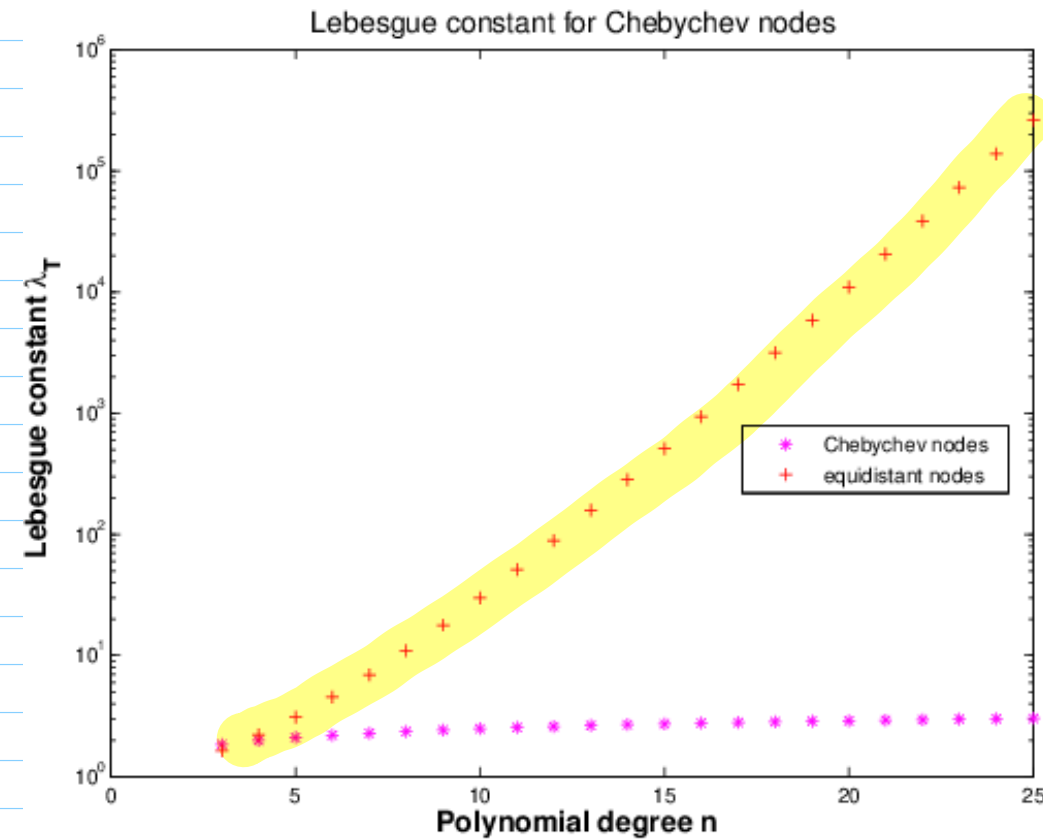
$$\|I_{\mathcal{T}}\|_{2 \rightarrow 2} := \sup_{\mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\|I_{\mathcal{T}}(\mathbf{y})\|_{L^2(I)}}{\|\mathbf{y}\|_2} \leq \left(\sum_{i=0}^n \|L_i\|_{L^2(I)}^2 \right)^{\frac{1}{2}}. \quad (5.2.4.12)$$

depends on $\mathcal{T}$

Lebesgue constant of $\mathcal{T}$: $\lambda_{\mathcal{T}} := \left\| \sum_{i=0}^n |L_i| \right\|_{L^\infty(I)} = \|I_{\mathcal{T}}\|_{\infty \rightarrow \infty}$

Rem. 5.2.4.12 : Lebesgue constant of **equidistant nodes**

$$\text{node set } \tau_n = \left\{ -1 + 2 \frac{k}{n} \right\}_{k=0}^n \subset I := [-1, 1]$$



$\triangle$ lin-log plot

$$\lambda_{\tau_n} \geq c e^{n/2}$$

exp. growth



Due to potentially "high sensitivity" interpolation with global polynomials of high degree is not suitable for data interpolation.

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▷ Review questions 5.2.4.16

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