

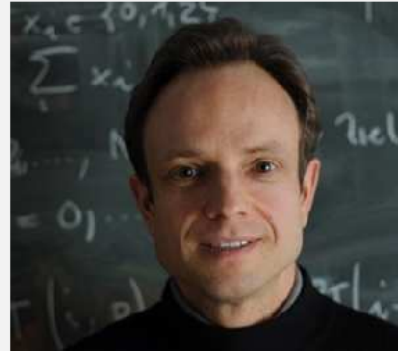
Course Video

Section 5.3: Shape-Preserving Interpolation

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Notion of monotonicity and convexity of functions

Dependencies. Section 5.1, Section 5.2.1

Duration: 47 minutes

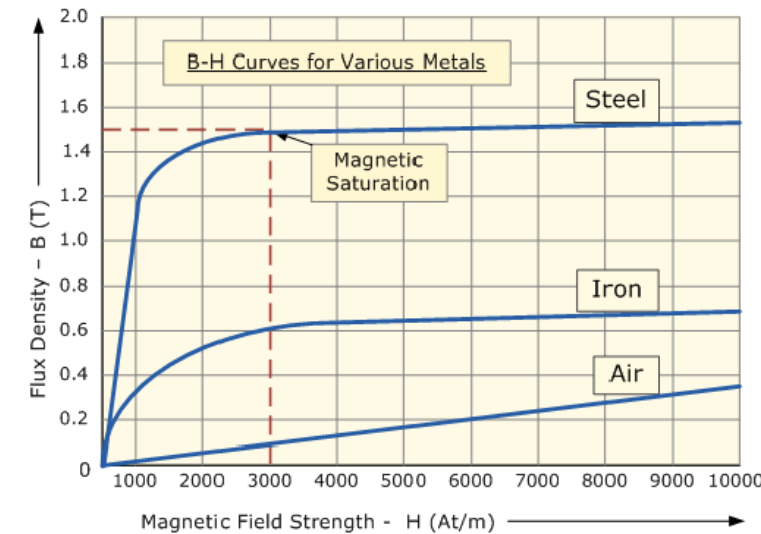
V. Data Interpolation and Data Fitting in 1D

5.3. Shape-Preserving Interpolation

Ex 5.3.0.1. (Magnetization curves)

- ◆ $H \mapsto B(H)$ smooth (at least C^1),
- ◆ $H \mapsto B(H)$ monotonic (increasing),
- ◆ $H \mapsto B(H)$ concave [curvature]

↑
shape properties



②

5.3.1. Shape properties of Functions & Data

§ 5.3.1.1. The "shape" of data

[Sorted nodes: $t_0 < t_1 < \dots < t_n$]

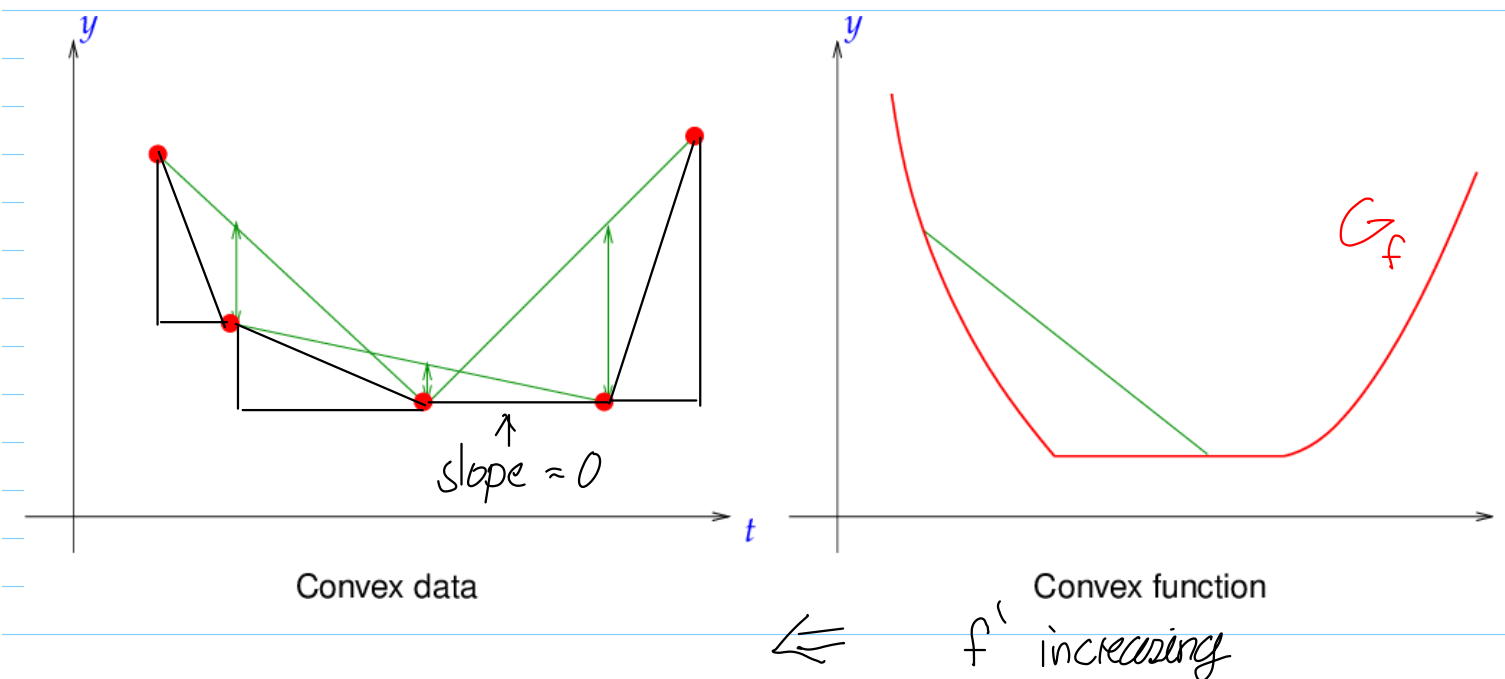
Definition 5.3.1.2. monotonic data

The data (t_i, y_i) are called **monotonic** when $y_i \geq y_{i-1}$ or $y_i \leq y_{i-1}$, $i = 1, \dots, n$.

"Curvature of data"

Definition 5.3.1.4. Convex/concave function $\rightarrow$

$f: I \subset \mathbb{R} \mapsto \mathbb{R}$ **convex** **concave** $\Leftrightarrow$ $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall 0 \leq \lambda \leq 1$
 $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in I.$



Definition 5.3.1.3. Convex/concave data

The data $\{(t_i, y_i)\}_{i=0}^n$ are called **convex** (**concave**) if

$$\Delta_j \stackrel{(\geq)}{\leq} \Delta_{j+1}, \quad j = 1, \dots, n-1, \quad \Delta_j := \frac{y_j - y_{j-1}}{t_j - t_{j-1}}, \quad j = 1, \dots, n.$$

monotonicity of slopes

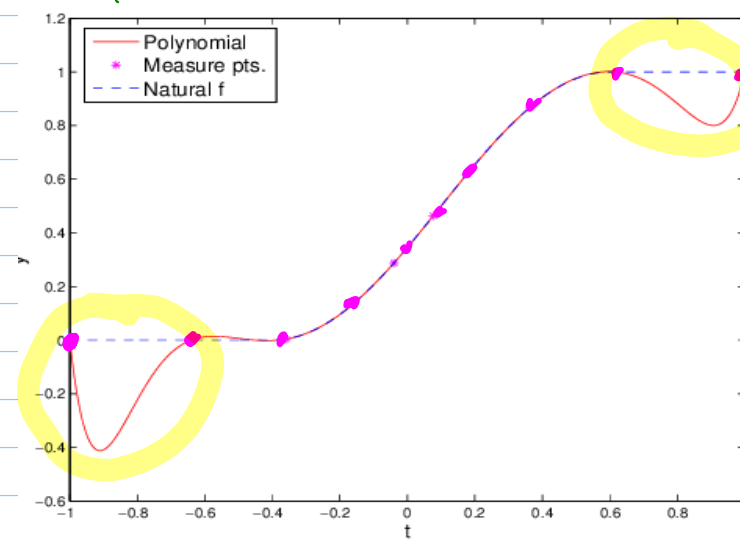
slopes

Local Shape preservation:

shape property of data $\Rightarrow$ same shape property of interpolation

hold in every interval $[t_j, t_k]$, $0 \leq j < k \leq n$

Exp.: 5.3.1.6 (Global polynomial interpolation)



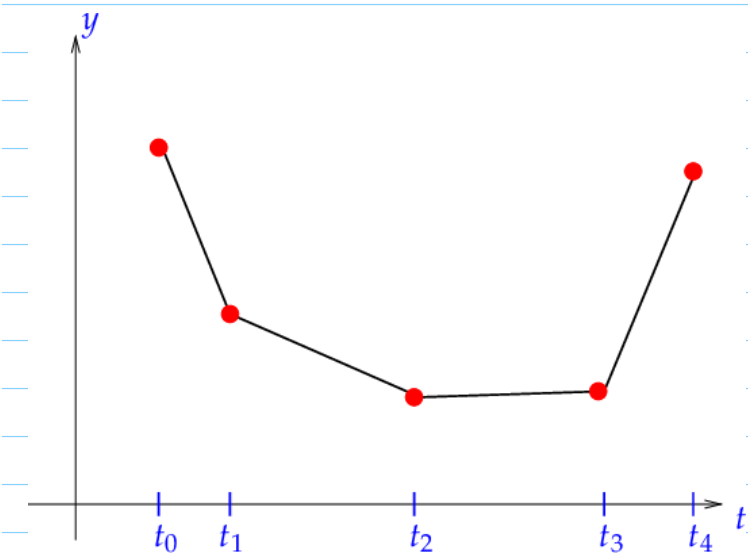
Δ Oscillations

$\leftarrow$ Interpolating polynomial, degree = 10

- no locality,
- no positivity,
- no monotonicity,
- no local conservation of the curvature,

③

5.3.2. Piecewise Linear Interpolation



△ polygon

- continuous interpolant
- local
- linear interpolation scheme
- perfectly locally shape-preserving

Theorem 5.3.2.3. Local shape preservation by piecewise linear interpolation

Let $s \in C([t_0, t_n])$ be the piecewise linear interpolant of $(t_i, y_i) \in \mathbb{R}^2$, $i = 0, \dots, n$, for every subinterval $I = [t_j, t_k] \subset [t_0, t_n]$:

- | | | |
|--|---------------|--|
| if $(t_i, y_i) _I$ are positive/negative | $\Rightarrow$ | $s _I$ is positive/negative, |
| if $(t_i, y_i) _I$ are monotonic (increasing/decreasing) | $\Rightarrow$ | $s _I$ is monotonic (increasing/decreasing), |
| if $(t_i, y_i) _I$ are convex/concave | $\Rightarrow$ | $s _I$ is convex/concave. |

perfectly local

A single flaw:

p.w. linear interpolant merely C^0 , but not C^1
(lack of smoothness)

5.3.3. Cubic Hermite Interpolation

Goal/policy:

- operates locally
 - use p.w. polynomials
 - achieve C^1 by raising degree
- like p.w. linear interp.
- degree = 3

5.3.3.1 Definition and Algorithms

Definition 5.3.3.1. Cubic Hermite polynomial interpolant

Given data points $(t_j, y_j) \in \mathbb{R} \times \mathbb{R}$, $j = 0, \dots, n$, with pairwise distinct ordered nodes t_j , and slopes $c_j \in \mathbb{R}$, the piecewise cubic Hermite interpolant $s : [t_0, t_n] \rightarrow \mathbb{R}$ is defined by the requirements

$$s|_{[t_{i-1}, t_i]} \in \mathcal{P}_3, \quad i = 1, \dots, n, \quad s(t_i) = y_i, \quad s'(t_i) = c_i, \quad i = 0, \dots, n.$$

piecewise polynomial

interpolation conditions

$$s \in C^1([t_0, t_n])$$

[because s' continuous by construction]

④ Compute with local representations

► $s(t) = y_{i-1}H_1(t) + y_iH_2(t) + c_{i-1}H_3(t) + c_iH_4(t)$, $t \in [t_{i-1}, t_i]$, (5.3.3.4)

$H_i \in \mathcal{P}_3$:

a generalized cardinal basis

	$H(t_{i-1})$	$H(t_i)$	$H'(t_{i-1})$	$H'(t_i)$
H_1	1	0	0	0
H_2	0	1	0	0
H_3	0	0	1	0
H_4	0	0	0	1

↑ by chain rule

$$H_1(t) := \phi\left(\frac{t_i - t}{h_i}\right), \quad (5.3.3.5a)$$

$$H_2(t) := \phi\left(\frac{t - t_{i-1}}{h_i}\right), \quad (5.3.3.5b)$$

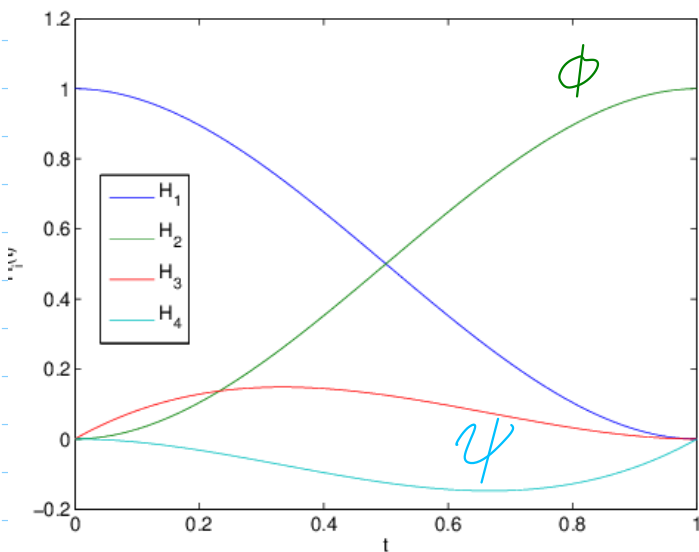
$$H_3(t) := -h_i\psi\left(\frac{t_i - t}{h_i}\right), \quad (5.3.3.5c)$$

$$H_4(t) := h_i\psi\left(\frac{t - t_{i-1}}{h_i}\right), \quad (5.3.3.5d)$$

$$h_i := t_i - t_{i-1}, \quad (5.3.3.5e)$$

$$\phi(\tau) := 3\tau^2 - 2\tau^3, \quad (5.3.3.5f)$$

$$\psi(\tau) := \tau^3 - \tau^2. \quad (5.3.3.5g)$$



Local basis polynomial on $[0, 1]$

Ex 5.3.3.9 (Average-based p.w. cubic Hermite interpolation)

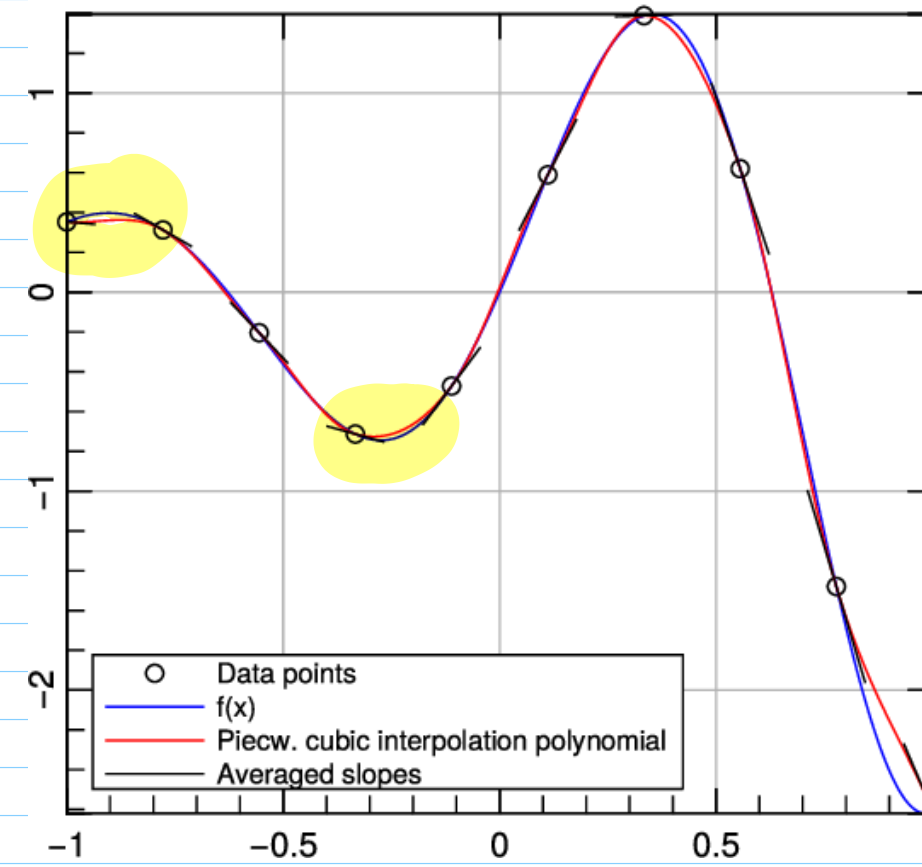
Needed: Reconstruction of slopes

Natural: use averages of data slopes

$$c_i = \begin{cases} \Delta_1 & \text{for } i = 0, \\ \Delta_n & \text{for } i = n, \\ \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} \Delta_i + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} \Delta_{i+1} & \text{if } 1 \leq i < n. \end{cases}, \quad \Delta_j := \frac{y_j - y_{j-1}}{t_j - t_{j-1}}, j = 1, \dots, n. \quad (5.3.3.8)$$

► linear interpolation scheme

Hermite interpolation



Not bad,
but not perfect!

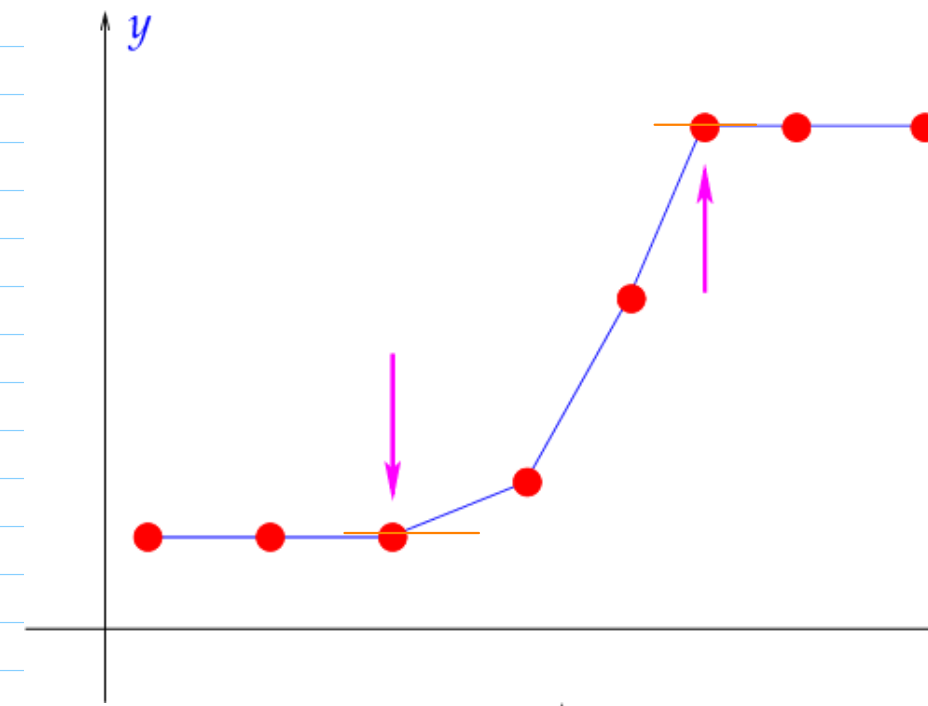
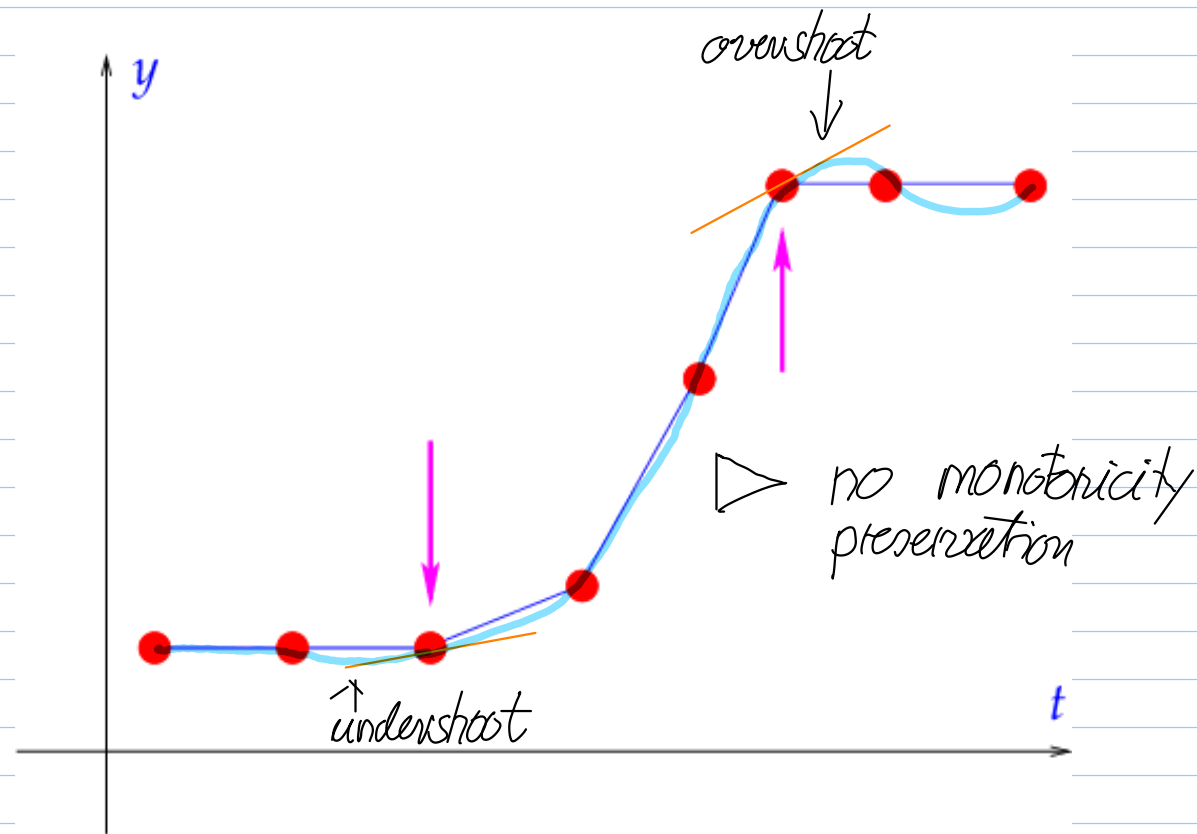
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5.3.3.2. Local Monotonicity-Preserving Hermite Interpolation

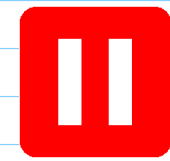
▷ Reconstruct slopes c_i that ensure local monotonicity preservation

The flaw of averaging:

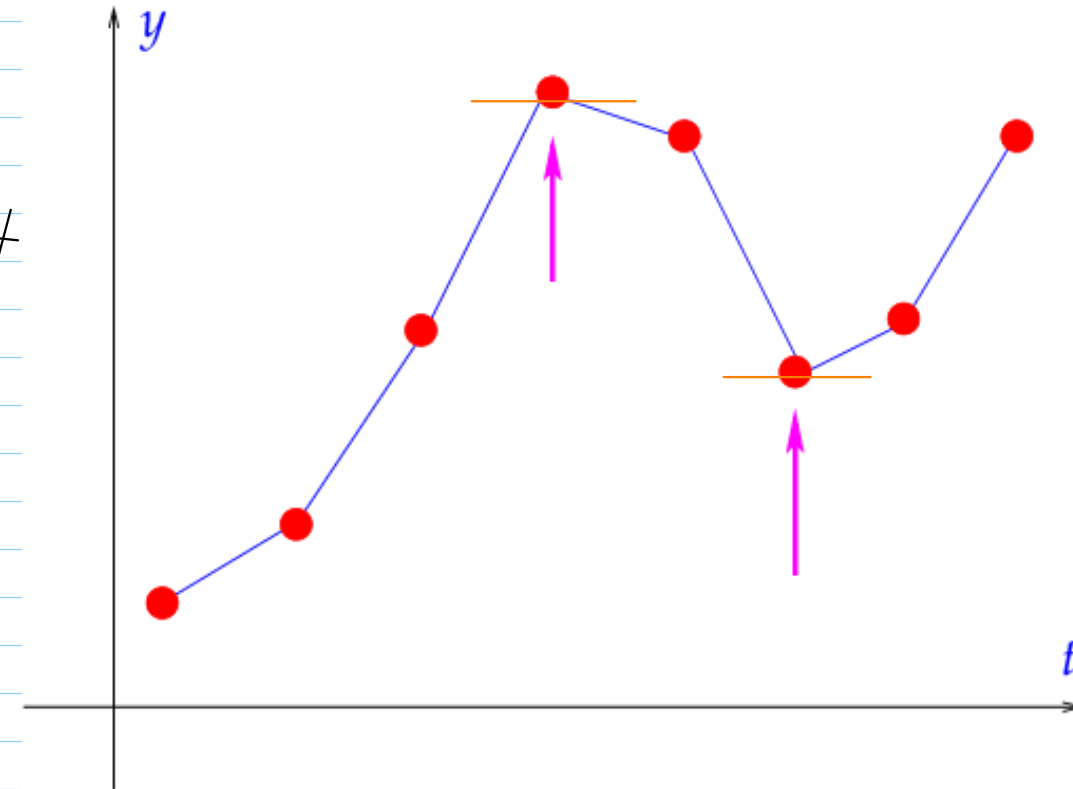
$$c_i = \begin{cases} \Delta_1 & , \text{ for } i = 0, \\ \Delta_n & , \text{ for } i = n, \\ \frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}\Delta_i + \frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}}\Delta_{i+1} & , \text{ if } 1 \leq i < n. \end{cases} \quad \Delta_j := \frac{y_j - y_{j-1}}{t_j - t_{j-1}}, j = 1, \dots, n. \quad (5.3.3.8)$$



Necessary for monotonicity preservation
slopes @ $\uparrow$ -points
= 0



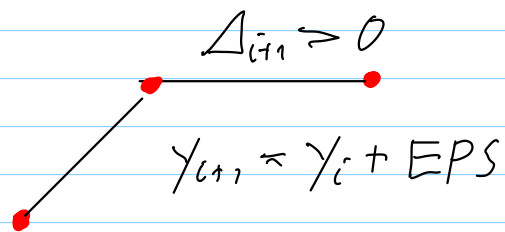
For monotonicity preservation:
 C' -interpolant must be flat at all interior extremal points!



⑥ Necessary: *Slope Limiting*

$$c_i = \begin{cases} 0 & \text{if } \text{sgn}(\Delta_i) \neq \text{sgn}(\Delta_{i+1}), \\ \text{some "average" of } \Delta_i, \Delta_{i+1} & \text{otherwise} \end{cases}, \quad i = 1, \dots, n-1. \quad (5.3.3.11)$$

notation: sign function $\text{sgn}(\xi) = \begin{cases} 1 & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \\ -1 & \text{if } \xi < 0. \end{cases}$



$\triangleright \text{average}(\Delta_i, \Delta_{i+1}) \approx 0$, if $\Delta_i \approx 0$ or $\Delta_{i+1} \approx 0$

Candidates:

(1) Geometric mean: $c_i = \sqrt{\Delta_i \Delta_{i+1}}$

$\rightarrow$ Observation: not perfectly monotonically pres.

(2) [Weighted] harmonic mean

$$c_i = \frac{1}{\frac{w_a}{\Delta_i} + \frac{w_b}{\Delta_{i+1}}}, \rightarrow 0 \quad (5.3.3.12)$$

with weights $w_a > 0, w_b > 0, (w_a + w_b = 1)$. as $\Delta_i \rightarrow 0$ or $\Delta_{i+1} \rightarrow 0$

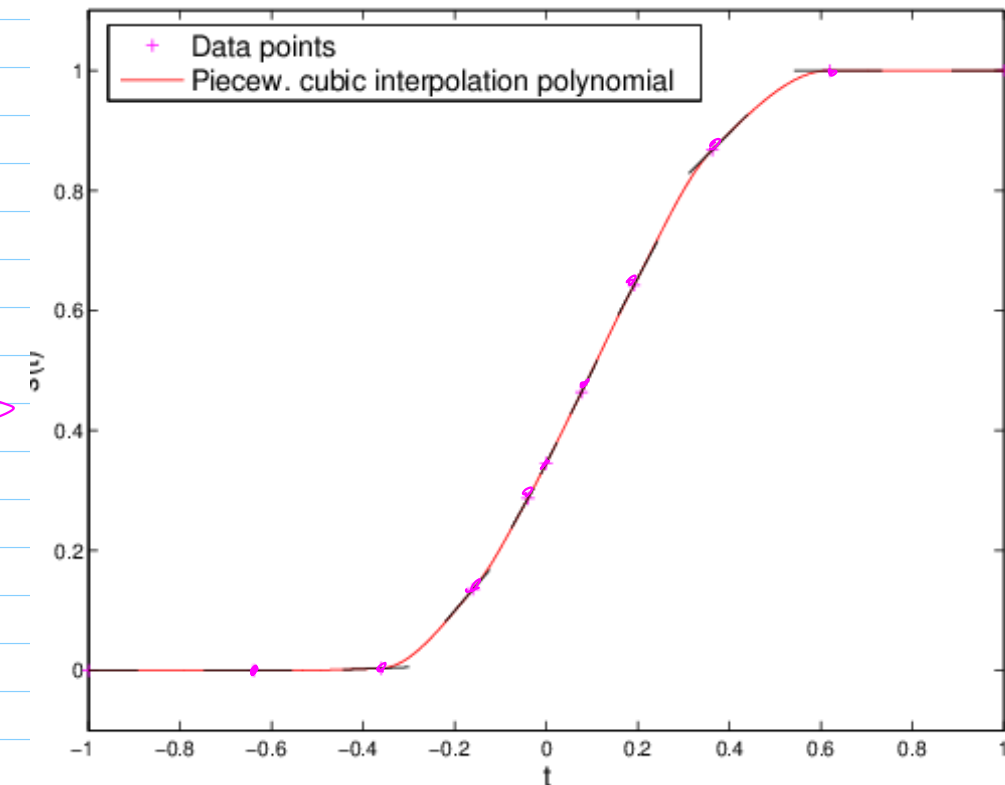
"pchip"

$$\text{sgn}(\Delta_1) = \text{sgn}(\Delta_2) \rightarrow c_i = \begin{cases} \Delta_1 & \text{if } i = 0, \\ \frac{3(h_{i+1} + h_i)}{2h_{i+1} + h_i + 2h_i + h_{i+1}} & \text{for } i \in \{1, \dots, n-1\}, \\ \Delta_n & \text{if } i = n, \end{cases} \quad h_i := t_i - t_{i-1}. \quad (5.3.3.13)$$

Theorem 5.3.3.17. Monotonicity preservation of limited cubic Hermite interpolation

The cubic Hermite interpolation polynomial with slopes as in Eq. (5.3.3.13) provides a local monotonicity-preserving C^1 -interpolant.

Ex. 5.3.3.14



$\rightarrow$ Not a linear interpolation operator

⑦

▷ Review questions 5.3.3.19