

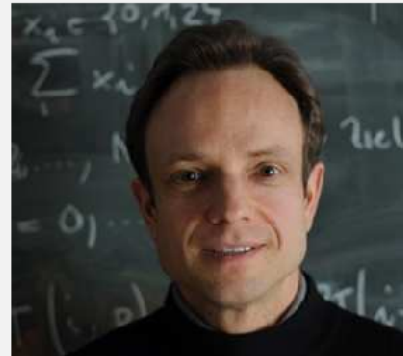
Course Video

Section 5.4.2: Cubic Spline Interpolation

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

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Dependencies.

Duration: 26 minutes

V. Data Interpolation and Data Fitting in 1D

5.4 Splines

5.4.2. Cubic-Spline Interpolation (CSI)

One-dimensional interpolation

Given: data points (t_i, y_i) , $i = 0, \dots, n$, $n \in \mathbb{N}$, $t_i \in I \subset \mathbb{R}$, $y_i \in \mathbb{R}$ Objective: Reconstruction of a function $f : I \mapsto \mathbb{R}$

- satisfying the $n + 1$ **interpolation conditions** (IC)

$$f(t_i) = y_i, \quad i = 0, \dots, n.$$

(5.1.0.2)

- and belonging to a set V of eligible functions.

The function f we find is called the **interpolant** of the given data set $\{(t_i, y_i)\}_{i=0}^n$.

Now: $V = S_{3,M} \subset C^2([t_0, t_n])$
 with $M = \{t_0 < t_1 < \dots < t_n\}$
 node = knots

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▷ Generalizes p.w. linear interpolation: $V = S_{1,M}$

Definition 5.4.2.3. Cubic-spline interpolant

Given a node set/knot set $\mathcal{M} := \{t_0 < t_1 < \dots < t_n\}$, $n \in \mathbb{N}$, and data values $y_0, \dots, y_n \in \mathbb{R}$, an associated **cubic spline interpolant** is a function $s \in S_{3,M}$ that complies with the interpolation conditions

$$|C : \quad s(t_j) = y_j, \quad j = 0, \dots, n. \quad (5.4.2.4)$$

▷ $\dim S_{3,M} = 3 + n \iff n+1$ I.C.
 → 2 excess d.o.f.s

§ 5.4.2.5 Computing cubic-spline interpolants

Use local representation ($\iff$ CHI)

(5.3.3.4) $\blacktriangleright$

$$s_{|[t_{j-1}, t_j]}(t) = \begin{matrix} s(t_{j-1}) & \cdot & (1 - 3\tau^2 + 2\tau^3) & + \\ s(t_j) & \cdot & (3\tau^2 - 2\tau^3) & + \\ h_j s'(t_{j-1}) & \cdot & (\tau - 2\tau^2 + \tau^3) & + \\ h_j s'(t_j) & \cdot & (-\tau^2 + \tau^3), \end{matrix} \quad (5.4.2.4)$$

with $h_j := t_j - t_{j-1}$, $\tau := (t - t_{j-1})/h_j$.

$\hat{=}$ known values $y_i = s(t_i)$

$\hat{=}$ slopes $s'(t_i)$: unknown(s) c_i

▷ ensures I.C., s p.w. $\mathcal{P}_3$, $s \in C^1$

To be imposed: C^2 -continuity **only**

$s''_{|[t_{i-1}, t_i]}(t_i) = s''_{|[t_i, t_{i+1}]}(t_i)$, $i = 1, \dots, n-1$
 → $n-1$ linear equation

$$s''_{|[t_{j-1}, t_j]}(t) = \overset{\cancel{y_{j-1}}}{s(t_{j-1})} h_j^{-2} 6(-1 + 2\tau) + \overset{\cancel{y_j}}{s(t_j)} h_j^{-2} 6(1 - 2\tau) + h_j^{-1} \overset{\cancel{c_{j-1}}}{s'(t_{j-1})} (-4 + 6\tau) + h_j^{-1} \overset{\cancel{c_j}}{s'(t_j)} (-2 + 6\tau), \quad \tau := (t - t_{j-1})/h_j, \quad (5.4.2.8)$$

Eq. (5.4.2.8) $\Rightarrow$

$$\begin{aligned} s''_{|[t_{j-1}, t_j]}(t_{j-1}) &= -6 \cdot s(t_{j-1}) h_j^{-2} + 6 \cdot s(t_j) h_j^{-2} - 4 \cdot h_j^{-1} s'(t_{j-1}) - 2 \cdot h_j^{-1} s'(t_j), \\ s''_{|[t_{j-1}, t_j]}(t_j) &= 6 \cdot s(t_{j-1}) h_j^{-2} - 6 \cdot s(t_j) h_j^{-2} + 2 \cdot h_j^{-1} s'(t_{j-1}) + 4 \cdot h_j^{-1} s'(t_j). \end{aligned}$$

$c_i := s'(t_i)$, $i = 0, \dots, n$

$$\frac{1}{h_j} c_{j-1} + \left(\frac{2}{h_j} + \frac{2}{h_{j+1}} \right) c_j + \frac{1}{h_{j+1}} c_{j+1} = 3 \left(\frac{y_j - y_{j-1}}{h_j^2} + \frac{y_{j+1} - y_j}{h_{j+1}^2} \right), \quad (5.4.2.9)$$

for $j = 1, \dots, n-1$.

▷ $(n-1) \times (n+1)$ underdetermined LSE
 2 "missing" equation

③

$$\begin{bmatrix} b_0 & a_1 & b_1 & 0 & \dots & \dots & 0 \\ 0 & b_1 & a_2 & b_2 & & & \\ & 0 & \ddots & \ddots & \ddots & & \\ \vdots & & & & & & \\ & & & & & a_{n-2} & b_{n-2} & 0 \\ 0 & \dots & \dots & 0 & b_{n-2} & a_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 3\left(\frac{y_1 - y_0}{h_1^2} + \frac{y_2 - y_1}{h_2^2}\right) \\ \vdots \\ 3\left(\frac{y_{n-1} - y_{n-2}}{h_{n-1}^2} + \frac{y_n - y_{n-1}}{h_n^2}\right) \end{bmatrix}. \quad (5.4.2.10)$$

with

$$b_i := \frac{1}{h_{i+1}}, \quad a_i := \frac{2}{h_i} + \frac{2}{h_{i+1}}, \quad i = 0, 1, \dots, n-1, \quad [b_i, a_i > 0, \quad a_i = 2(b_i + b_{i-1})].$$

Additional constraint to ensure uniqueness of CSI

① Complete CSI

Fix $c_0 := s'(t_0)$, $c_n := s'(t_n) \rightarrow$ remove 2 unknowns
 $\rightarrow$ move to r.h.s.

$$\begin{bmatrix} b_0 & a_1 & b_1 & 0 & \dots & \dots & 0 \\ 0 & b_1 & a_2 & b_2 & & & \\ & 0 & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & & a_{n-2} & b_{n-2} & 0 \\ 0 & \dots & \dots & 0 & b_{n-2} & a_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 3\left(\frac{y_1 - y_0}{h_1^2} + \frac{y_2 - y_1}{h_2^2}\right) - c_0 b_0 \\ \vdots \\ 3\left(\frac{y_{n-1} - y_{n-2}}{h_{n-1}^2} + \frac{y_n - y_{n-1}}{h_n^2}\right) - c_n b_{n-1} \end{bmatrix}. \quad (5.4.2.10)$$

with

$\rightarrow (n-1) \times (n-1)$ LSE

$$b_i := \frac{1}{h_{i+1}}, \quad a_i := \frac{2}{h_i} + \frac{2}{h_{i+1}}, \quad i = 0, 1, \dots, n-1, \quad [b_i, a_i > 0, \quad a_i = 2(b_i + b_{i-1})].$$

② Natural CSI : $s'(t_0) = s'(t_n) = 0$

$\rightarrow$ two extra equations

$$\frac{2}{h_1} c_0 + \frac{1}{h_1} c_1 = 3 \frac{y_1 - y_0}{h_1^2}, \quad \frac{1}{h_n} c_{n-1} + \frac{2}{h_n} c_n = 3 \frac{y_n - y_{n-1}}{h_n^2}.$$

$\triangleright (n+1) \times (n+1)$ LSE

③ Periodic CSI

$$c_0 = c_n \rightarrow -1 \text{ unknown}$$

$$s''(t_0) = s''(t_n) \rightarrow +1 \text{ equation}$$

$\triangleright n \times n$ LSE

$\triangleright$ Review questions 5.4.2.16

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