

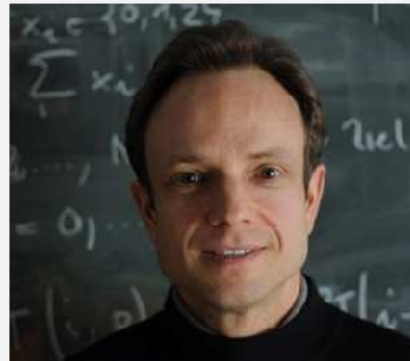
Course Video

Section 5.4.3: Structural Properties of Cubic Spline Interpolants

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Dependencies. Section 5.4.2

Duration: 24 minutes

V. Data Interpolation and Data Fitting in 1D

5.4 Splines

5.4.3. Structural Properties of Cubic-Spline Interpolants

Definition 5.4.2.3. Cubic-spline interpolant

Given a node set/knot set $\mathcal{M} := \{t_0 < t_1 < \dots < t_n\}$, $n \in \mathbb{N}$, and data values $y_0, \dots, y_n \in \mathbb{R}$, an associated **cubic spline interpolant** is a function $s \in \mathcal{S}_{3,\mathcal{M}}$ that complies with the interpolation conditions

$$s(t_j) = y_j \quad , \quad j = 0, \dots, n. \quad (5.4.2.4)$$

▷ Non-unique

Unique natural C SI : $s''(t_0) = s''(t_n) = 0$

②

§5.4.3.1 (Minimal bending energy of *natural* CSI)

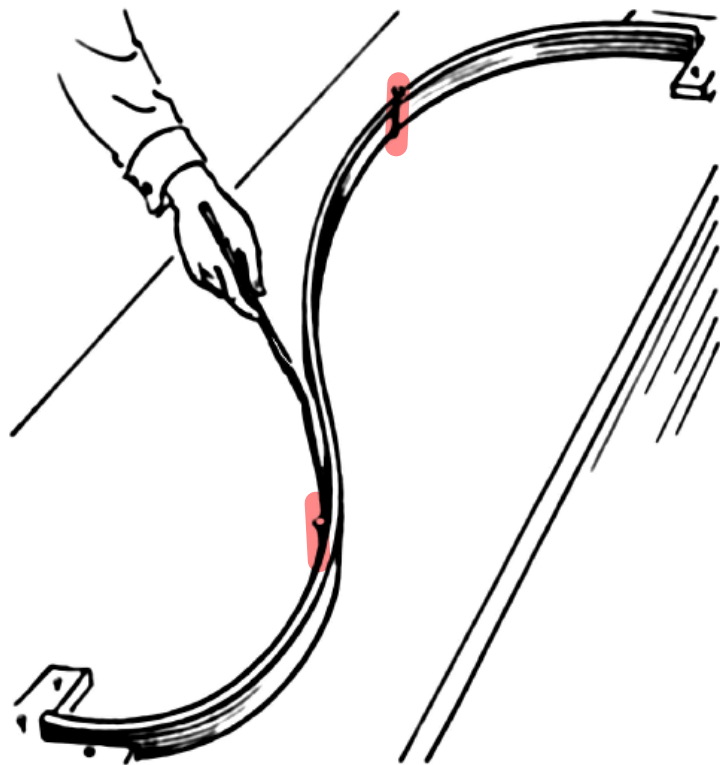
$$E_{bd}(f) = \frac{1}{2} \int_{t_0}^{t_n} |f''(t)|^2 dt$$

Theorem 5.4.3.3. Optimality of natural cubic spline interpolant

Given a knot/nodes set $\mathcal{M} := \{a = t_0 < t_1 < \dots < t_n = b\}$ in the interval $[a, b]$, let $s \in \mathcal{S}_{3, \mathcal{M}}$ be the *natural* cubic spline interpolant of the data points $(t_i, y_i) \in \mathbb{R}^2, i = 0, \dots, n$.

Then s minimizes the elastic bending energy *among all interpolating functions in $C^2([a, b])$* , that is

$$E_{bd}(s) \leq E_{bd}(f) \quad \forall f \in C^2([a, b]), f(t_i) = y_i, i = 0, \dots, n.$$



△ Natural CSI
= equilibrium shape
of "interpolating"
elastic rod

"Calculus of variations" proof
 $s \hat{=}$ natural CSI

perturbation: $k \in C^2([a, b]), k(t_i) = 0$
▷ $s + k$ yield any C^2 -interpolant

$$\begin{aligned} E_{bd}(s + k) &= \frac{1}{2} \int_a^b |s'' + k''|^2 dt \\ &= E_{bd}(s) + \underbrace{\int_a^b s''(t)k''(t) dt}_{:= I = 0} + \underbrace{\frac{1}{2} \int_a^b |k''|^2 dt}_{\geq 0} \geq E_{bd}(s) \end{aligned} \quad (5.4.3.4)$$

$$\begin{aligned} I &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} s''(t)k''(t) dt \\ &= \sum_{j=1}^n \left\{ - \int_{t_{j-1}}^{t_j} s'''(t)k'(t) dt + s''(t_j)k'(t_j) - s''(t_{j-1})k'(t_{j-1}) \right\} \quad [l.b.p.] \\ &= \sum_{j=1}^n \left\{ \int_{t_{j-1}}^{t_j} \underbrace{s^{(4)}(t)}_{=0} k(t) dt + s''(t_j)k'(t_j) - s''(t_{j-1})k'(t_j) + s'''(t_j^-)k(t_j) - s'''(t_{j-1}^+)k(t_{j-1}) \right\} \quad [i.b.p.] \\ &= - \sum_{j=1}^n \left(\underbrace{s'''(t_j^-)k(t_j)}_{=0} - \underbrace{s'''(t_{j-1}^+)k(t_{j-1})}_{=0} \right) + \underbrace{s''(t_n)k'(t_n)}_{=0} - \underbrace{s''(t_0)k'(t_0)}_{=0} = 0. \end{aligned}$$

by telescopic sum

③

§ 5.4.3.7 (Weak locality of CSI)

perturbing y_j affects interpolant only near t_j

Ex: p.w. linear intp.: changing $y_j \rightarrow$ interpolant changes only in $[t_{j-1}, t_{j+1}]$
(strict locality)*

* not satisfied for CSI

$I: \mathbb{R}^{n+1} \rightarrow C^0([t_0, t_n]) \cong$ linear interpolation scheme

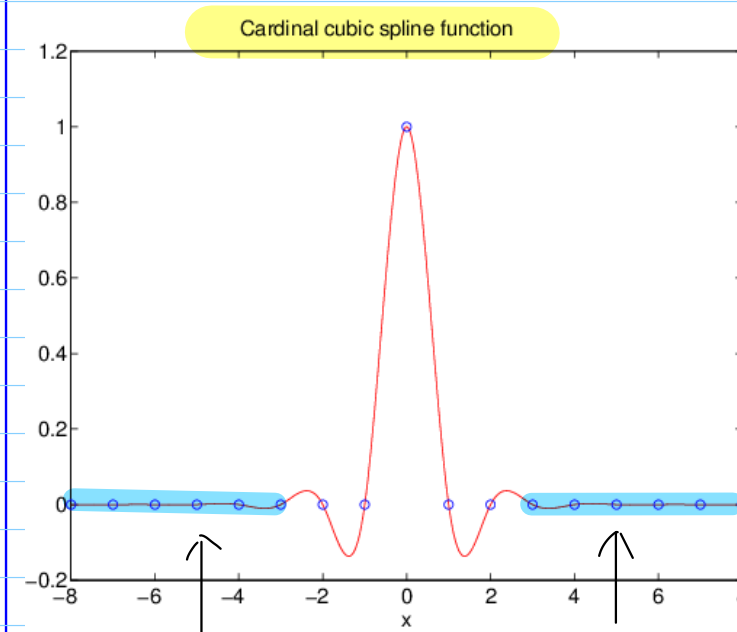
Locality of I can be seen from the

cardinal basis $\{b_j\}_{j=0}^n \subset V$

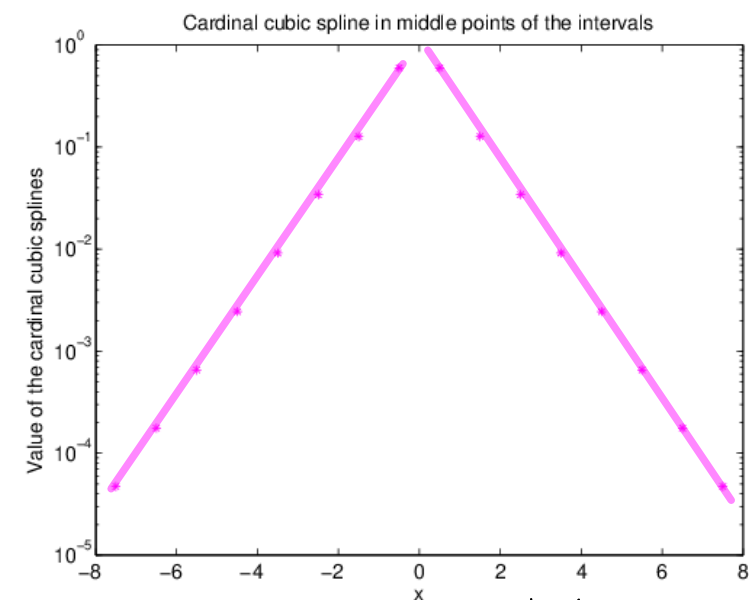
$$I(y) = \sum_{j=0}^n y_j b_j$$

▷ Perturbing y_j impacts $I(y)$ only inside the support of b_j : $\text{supp } b_j := \{t : b_j(t) \neq 0\}$

Exp 5.4.3.9: (Decay of cardinal splines relative to CSI)
[Equidistant nodes] $\rightarrow$ cardinal basis fct. of $S_{3,n}$



"virtually zero"



lin-log plot

▷ $b_j(t) \leq b_j(t_j) e^{-\lambda |t-t_j|}, \lambda > 0$
 $\cong$ exponential decay

④

▷ Review questions 5.4.3.11

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