

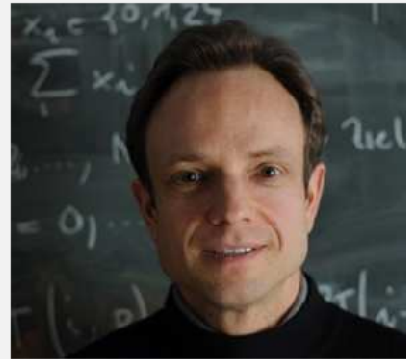
Course Video

Section 5.6: Trigonometric Interpolation

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Prerequisites.

- Complex arithmetic
- Relationship between trigonometric functions and the complex exponential function

Dependencies. Section 5.2.2, Section 5.2.3

Duration: 30 minutes

V. Data Interpolation and Data Fitting in 1D

5.6. Trigonometric Interpolation

Interpolate data points $(t_k, y_k) \in \mathbb{R}$, $k = 0, \dots, n$,
 obtained from sampling a **T -periodic*** function $f: \mathbb{R} \rightarrow \mathbb{R}$,
 $T > 0$ known

$$* f(t+T) = f(t) \quad \forall t \in \mathbb{R}$$

Assume: $0 \leq t_k < T$

▷ Want a T -periodic interpolant

Task: Given data points (t_i, y_i) , $y_i \in \mathbb{K}$, $t_i \in [0, T[$, find a **T -periodic** function $f: \mathbb{R} \rightarrow \mathbb{K}$ (the interpolant),
 $f(t+T) = f(t) \quad \forall t \in \mathbb{R}$, that satisfies the **interpolation conditions**

$$f(t_i) = y_i, \quad i = 0, \dots, n.$$

(5.6.0.2)

▷ Now: $T = 1$

5.6.1 Trigonometric Polynomials

Definition 5.6.1.1. Space of trigonometric polynomials

The vector space of 1-periodic **trigonometric polynomials** of degree $2n$, $n \in \mathbb{N}$, is given by

$$\mathcal{P}_{2n}^T := \text{Span}\{t \mapsto \cos(2\pi jt), t \mapsto \sin(2\pi jt)\}_{j=0}^n \subset C^\infty(\mathbb{R}).$$

↳ drop for $j=0$

Why polynomials:

$$(*) \quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$q(t) = \alpha_0 + \sum_{j=1}^n \alpha_j \cos(2\pi jt) + \beta_j \sin(2\pi jt)$$

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$$= \alpha_0 + \frac{1}{2} \left\{ \sum_{j=1}^n (\alpha_j - i\beta_j) e^{2\pi i j t} + (\alpha_j + i\beta_j) e^{-2\pi i j t} \right\} \quad (*)$$

$$= \alpha_0 + \frac{1}{2} \sum_{j=-n}^{-1} (\alpha_{-j} + i\beta_{-j}) e^{2\pi i j t} + \frac{1}{2} \sum_{j=1}^n (\alpha_j - i\beta_j) e^{2\pi i j t}$$

$$= e^{-2\pi i n t} \sum_{j=0}^{2n} \gamma_j e^{2\pi i j t}, \quad \text{with } \gamma_j := \begin{cases} \frac{1}{2}(\alpha_{n-j} + i\beta_{n-j}) & \text{for } j = 0, \dots, n-1, \\ \alpha_0 & \text{for } j = n, \\ \frac{1}{2}(\alpha_{j-n} - i\beta_{j-n}) & \text{for } j = n+1, \dots, 2n. \end{cases} \quad (5.6.1.4)$$

↑ $\in \mathbb{C}$

$$q \in \mathcal{P}_{2n}^T \Rightarrow q(t) = e^{-2\pi i n t} p(e^{2\pi i t})$$

with $p(z) = \sum_{j=0}^{2n} \gamma_j z^j$

Isomorphism: $\mathcal{P}_{2n} \longleftrightarrow \mathcal{P}_{2n}^T$

$$(\gamma_j \in \mathbb{C} \longleftrightarrow \alpha_j, \beta_j \in \mathbb{R})$$

Corollary 5.6.1.6. Dimension of $\mathcal{P}_{2n}^T$

The vector space $\mathcal{P}_{2n}^T$ has dimension $\dim \mathcal{P}_{2n}^T = 2n + 1$.

③

5.6.2 Reduction to Lagrange Interpolation

$$q \in \mathcal{P}_{2n}^T \Leftrightarrow q(t) = e^{-2\pi i n t} p(e^{2\pi i t}) \text{ for some } p \in \mathcal{P}_{2n}. \quad (5.6.2.2)$$

$$\text{I.C. : } q(t_k) = y_k \Leftrightarrow e^{-2\pi i n t_k} p(e^{2\pi i t_k}) = y_k$$

$$z_k := e^{2\pi i t_k}, \quad \tilde{y}_k = e^{2\pi i n t_k} y_k, \quad k=0, \dots, 2n$$

Trigonometric interpolation through data points $(t_k, y_k), k=0, \dots, 2n$ $\Leftrightarrow$ Lagrange polynomial interpolation $\in \mathbb{C}$ through data points $(e^{2\pi i t_k}, e^{2\pi i n t_k} y_k), k=0, \dots, 2n$

$\in \mathbb{C}^2$

▷ Existence & uniqueness of trig. interpolant

▷ Reuse functions for polynomial interpolation / evaluation.

5.6.3. Equidistant Trigonometric Interpolation

Now: data point $(t_k = \frac{k}{2n+1}, y_k) \in \mathbb{R}^2, k=0, \dots, 2n$

§ 5.6.3.1 : Computation of equidistant trigonometric interpolant

$$q \in \mathcal{P}_{2n}^T \Rightarrow q(t) = e^{-2\pi i n t} \cdot p(e^{2\pi i t}) \text{ with } p(z) = \sum_{j=0}^{2n} \gamma_j z^j \in \mathcal{P}_{2n}, \quad (5.6.3)$$

$$\text{I.C. : } q(t_k) = y_k$$

$$\sum_{j=0}^{2n} \gamma_j \exp(2\pi i \frac{jk}{2n+1}) = (\mathbf{b})_k := \exp(2\pi i \frac{nk}{2n+1}) y_k, \quad k=0, \dots, 2n. \quad (5.6.3.2)$$

$$\bar{\mathbf{F}}_{2n+1} \mathbf{c} = \mathbf{b}, \quad \mathbf{c} = [\gamma_0, \dots, \gamma_{2n}]^T$$

$(2n+1) \times (2n+1)$ (conjugate) Fourier matrix, see (4.2.1.13)!

Lemma 4.2.1.14

$$\bar{\mathbf{F}}_{2n+1} = \frac{1}{2n+1} \mathbf{F}_{2n+1}^T$$

$$\mathbf{c} = \frac{1}{2n+1} \mathbf{F}_{2n+1} \mathbf{b}. \quad (5.6.3.3)$$

$$\mathbf{F}_{2n+1} = [e^{-\frac{2\pi i}{2n+1} l k}]_{l,k=0}^{2n}$$

y_j by DFT_{2n+1} !

cost = $O(n \log n)$ for FFT-based implementation
[vs. cost = $O(n^3)$ for Gaussian elimination]

§ 5.6.3.6 Evaluation of equidistant trigonometric interpolant

Task: Find an efficient way to evaluate the trigonometric polynomial

$$q(t) = \alpha_0 + \sum_{j=1}^n \alpha_j \cos(2\pi jt) + \beta_j \sin(2\pi jt), \quad \alpha_j, \beta_j \in \mathbb{R}, \quad (5.6.1.3)$$

at **equidistant** points $\frac{k}{N}$, $N > 2n$. $k = 0, \dots, N-1$. [many point, $N \gg n$]
 $\uparrow$ given

$$q(t) = e^{-2\pi i n t} \sum_{j=0}^{2n} \gamma_j e^{2\pi i j t}, \quad \text{with } \gamma_j = \begin{cases} \frac{1}{2}(\alpha_{n-j} + i\beta_{n-j}) & \text{for } j = 0, \dots, n-1, \\ \alpha_0 & \text{for } j = n, \\ \frac{1}{2}(\alpha_{j-n} - i\beta_{j-n}) & \text{for } j = n+1, \dots, 2n. \end{cases}$$

$$q\left(\frac{k}{N}\right) = e^{-2\pi i n \frac{k}{N}} \underbrace{\sum_{j=0}^{2n} \gamma_j e^{2\pi i j \frac{k}{N}}}_{= (\overline{F_N \subseteq})_k}, \quad k = 0, \dots, N-1$$

$$(\subseteq)_j := \begin{cases} \gamma_j, & j \leq 2n \\ 0, & j > 2n \end{cases} \quad \uparrow \text{ via DFT}_N$$

$\triangleright$ Cost = $O(N \log N)$ for FFT-based impl., $N \rightarrow \infty$

$\triangleright$ Review questions 5.6.3.10

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