

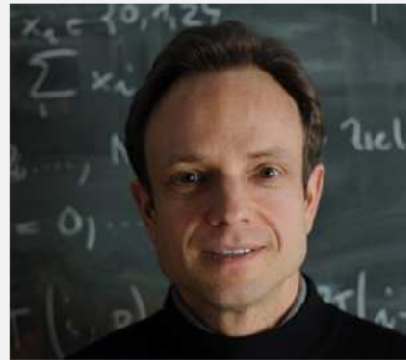
Course Video

Section 6.2: Polynomial Approximation: Theory

Prof. R. Hiptmair, SAM, ETH Zurich

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(C) Seminar für Angewandte Mathematik, ETH Zürich

**Prerequisites.**

- Elementary calculus of one variable

Dependencies. Section 5.2.1

Duration: 25 minutes

VI. Approximation of Functions in One Dimension

6.2. Approximation by Global Polynomials

Taylor polynomials: polynomials are "natural" for approximation

$$f(t) \approx \underbrace{f(t_0) + f'(t_0)(t - t_0) + \frac{f^{(2)}(t_0)}{2}(t - t_0)^2 + \dots + \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k}_{=: T_k(t) \in \mathcal{P}_k}, \text{ for some } t_0 \in I.$$

for $f \in C^k(I) \triangleq$ smoothness requirements

$\Rightarrow f \sim T_k$ locally

②

6.2.1. Polynomial Approximation: Theory

For "large enough" degree : polynomials can approximate anything

Theorem 6.2.1.2. Uniform approximation by polynomials

For $f \in C^0([0,1])$, define the n -th Bernstein approximant as

$$p_n(t) = \sum_{j=0}^n f(j/n) \binom{n}{j} t^j (1-t)^{n-j}, \quad p_n \in \mathcal{P}_n. \quad (6.2.1.3)$$

It satisfies $\|f - p_n\|_\infty \rightarrow 0$ for $n \rightarrow \infty$. If $f \in C^m([0,1])$, then even $\|f^{(k)} - p_n^{(k)}\|_\infty \rightarrow 0$ for $n \rightarrow \infty$ and all $0 \leq k \leq m$.

☹ no further information

More smoothness $\Rightarrow$ better estimate

Theorem 6.2.1.11. L^∞ polynomial best approximation estimate [Jackson thm.]

If $f \in C^r([-1,1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then, for any polynomial degree $n \geq r$,

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} \leq (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \|f^{(r)}\|_{L^\infty([-1,1])},$$

where

$$\|f^{(r)}\|_{L^\infty([-1,1])} := \max_{x \in [-1,1]} |f^{(r)}(x)|.$$

Max-norm of best-approximation error

A simpler bound : reveals asymptotics for $n \rightarrow \infty$

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} \leq C(r) n^{-r} \|f^{(r)}\|_{L^\infty([-1,1])}, \quad (6.2.1.13)$$

$\uparrow$ > 0 , independent of f and n

$$(6.2.1.13) \Rightarrow \inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} = O(n^{-r}) \text{ for } n \rightarrow \infty.$$

Algebraic convergence with rate r for $n \rightarrow \infty$.

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§§ 6.2.1.14 (Transformation of approximation error estimates)

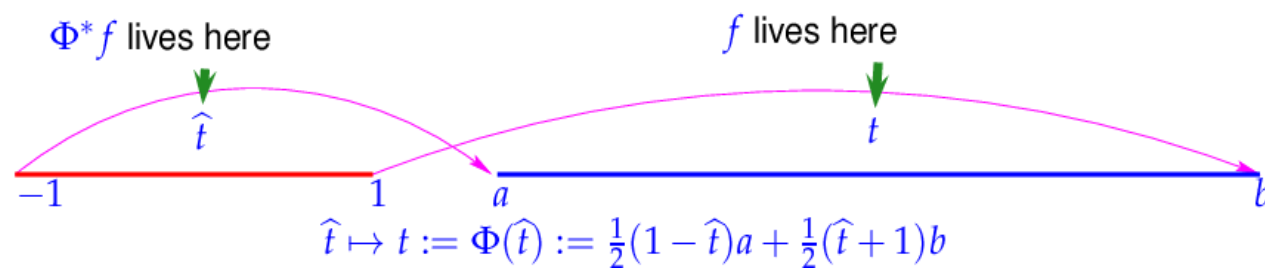
Jackson thm. on $[a, b]$?

Tool: Bijective affine transformation

$$\Phi: [-1, 1] \rightarrow [a, b], \quad \Phi(\hat{t}) := a + \frac{1}{2}(\hat{t} + 1)(b - a), \quad -1 \leq \hat{t} \leq 1, \quad (6.2.1.15)$$

▷ Pullback of functions

$$\Phi^*: C^0([a, b]) \rightarrow C^0([-1, 1]), \quad \Phi^*(f)(\hat{t}) := (f \circ \Phi)(\hat{t}) := f(\Phi(\hat{t})), \quad -1 \leq \hat{t} \leq 1. \quad (6.2.1.16)$$



$$\| \Phi^* f \|_{\infty, [-1, 1]} = \| f \|_{\infty, [a, b]}$$

$$p \in \mathcal{P}_n \Rightarrow \Phi^* p \in \mathcal{P}_n$$

Pullback & derivatives ($\Phi \leftrightarrow$ dilation)

$$\text{Chain rule: } \frac{d}{d\hat{t}}(\Phi^* f)(\hat{t}) \stackrel{\text{C.R.}}{=} \frac{df}{dt}(\Phi(\hat{t})) \frac{d\Phi}{d\hat{t}} = \frac{df}{dt}(\Phi(\hat{t})) \cdot \frac{1}{2}(b - a),$$

$$(\Phi^* f)^{(r)} = \left(\frac{b-a}{2} \right)^r \Phi^*(f^{(r)}). \quad (6.2.1.24)$$

Lemma 6.2.1.20

$$\|(\Phi^* f)^{(r)}\|_{L^\infty([-1, 1])} = \left(\frac{b-a}{2} \right)^r \|f^{(r)}\|_{L^\infty([a, b])}, \quad f \in C^r([a, b]), r \in \mathbb{N}_0. \quad (6.2.1.25)$$

§ 6.2.1.26 (Polynomial best-approximation error estimates on general intervals)

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([a, b])} \stackrel{(*)}{=} \inf_{p \in \mathcal{P}_n} \|\Phi^* f - p\|_{L^\infty([-1, 1])} \quad [\Phi^* \text{ preserves } \mathcal{P}_n!]$$

Jackson thm

→ Thm. 6.2.1.11

$$\leq (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \|(\Phi^* f)^{(r)}\|_{L^\infty([-1, 1])}$$

$$\stackrel{(6.2.1.24)}{=} (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \left(\frac{b-a}{2} \right)^r \|f^{(r)}\|_{L^\infty([a, b])}$$

↑
length of interval

(4)

§ 6.2.1.14 (Transformation of polynomial approximation schemes)

Known: Approximation scheme $\hat{A} : C^0([-1,1]) \rightarrow \mathcal{P}_n$

Define: $A := (\phi^{-1})^* \circ \hat{A} \circ \phi^* : C^0([a,b]) \rightarrow \mathcal{P}_n$
 = approximation scheme on $[a,b]$

