

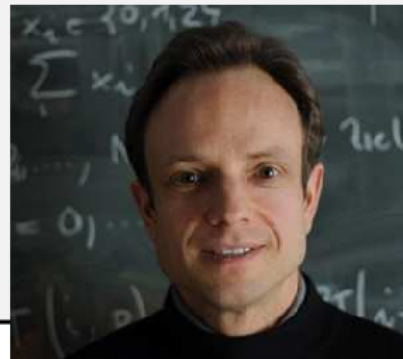
Course Video

Section 6.2.2.2: Error Estimates for Polynomial Interpolation: Interpolands of Finite Smoothness

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Calculus in 1 variable
- Mean-value theorem in 1D

Dependencies. Section 6.2.2.1, Section 5.2.4

Duration: 30 minutes

VI. Approximation of Functions in One Dimension

6.2. Approximation by Global Polynomials

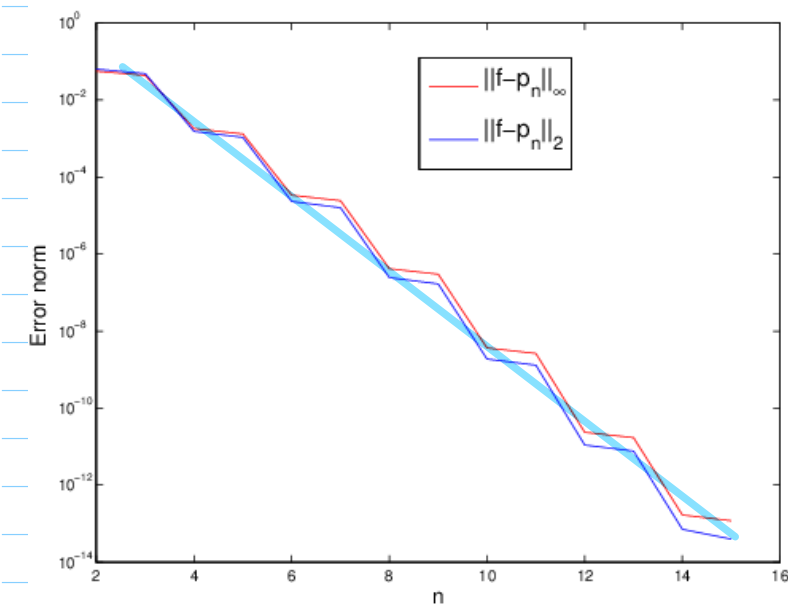
6.2.2. Error Estimates for Polynomial Interpolation

Exp. 6.2.2.4 : (Asymptotic behavior of Lagrange interpolation error)

$$f(t) := \sin t, \quad I = [0, \pi], \quad \mathcal{I}_n = \left\{ \frac{\pi}{n} j \right\}_{j=0}^n$$

↑
equid. nodes

2

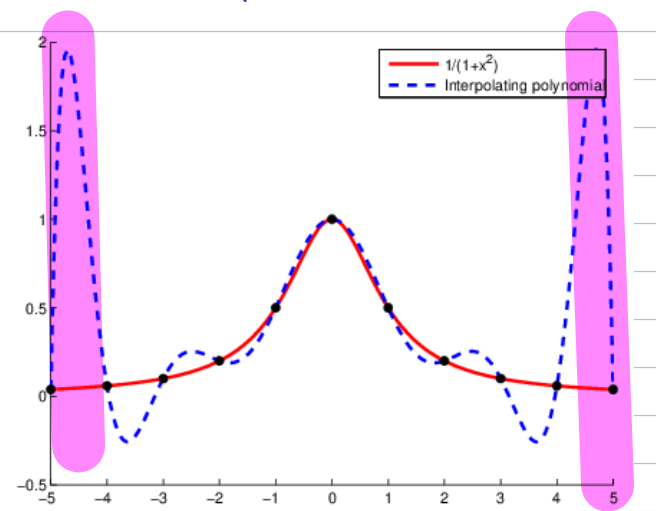


$$\triangle \|f - L_{\mathcal{T}_n} f\|_{\infty} \text{ and } \|f - L_{\mathcal{T}_n} f\|_2$$

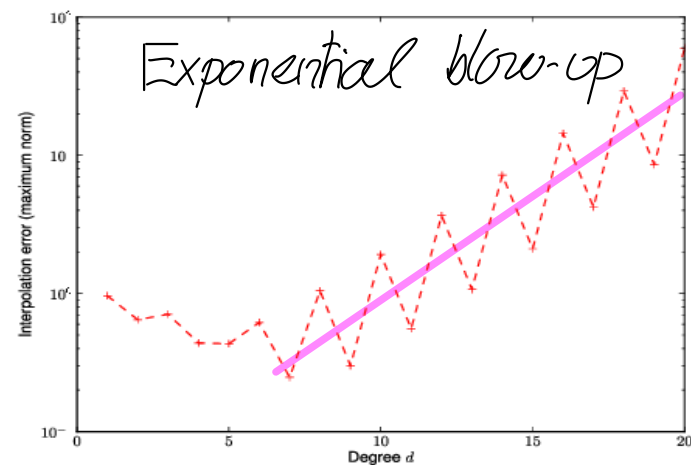
$\triangle$ Exponential convergence

Ex. 6.2.2.11 (Runge's counterexample) $\downarrow$ equid. nodes

$$f(t) = \frac{1}{1+t^2}, \quad I = [-5, 5], \quad \mathcal{T}_n = \left\{ -5 + \frac{10}{n} j \right\}_{j=0}^n$$



Interpolating polynomial, $n = 10$



Approximate $\|f - L_{\mathcal{T}_n} f\|_{\infty}$ on $[-5, 5]$

6.2.2.2. Interpolands of Finite Smoothness

$\hookrightarrow$ class C^k

Main theoretical result:

Theorem 6.2.2.15. Representation of interpolation error

We consider $f \in C^{n+1}(I)$ and the Lagrangian interpolation approximation scheme ($\rightarrow$ Def. 6.2.2.1) for a node set $\mathcal{T} := \{t_0, \dots, t_n\} \subset I$. Then, $\rightarrow$ not known

for every $t \in I$ there exists a $\tau_t \in [\min\{t, t_0, \dots, t_n\}, \max\{t, t_0, \dots, t_n\}]$ such that

$$\underbrace{f(t) - L_{\mathcal{T}}(f)(t)}_{\text{interpolation error}} = \frac{f^{(n+1)}(\tau_t)}{(n+1)!} \cdot \underbrace{\prod_{j=0}^n (t - t_j)}_{=: W \triangleq \text{nodal polynomial}}. \quad (6.2.2.16)$$

$W|_{\mathcal{T}} = 0$

Proof: Fix $t \in I \setminus \mathcal{T}$

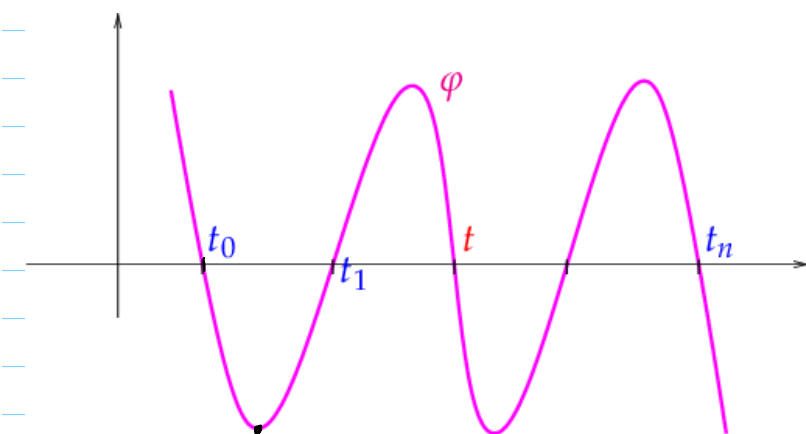
Choose $c \in \mathbb{R}$: $f(t) - L_{\mathcal{T}} f(t) - \underbrace{c W(t)}_{\neq 0} = 0$

$$\varphi(x) := f(x) - L_{\mathcal{T}} f(x) - c W(x) \in C^{n+1}$$

$$\varphi(t_j) = 0 \text{ by I.C.} \quad \& \quad \varphi(t) = 0 \text{ by choice of } c$$

$\triangleright \varphi$ has (at least) $n+2$ zeros

③


 φ' has at least $n+1$ zeros

 $\varphi^{(k)}$ has at least $n+2-k$ distinct zeros

 $\varphi^{(n+1)}$ has a zero τ_t

$$\varphi^{(n+1)}(x) = f^{(n+1)}(x) - 0 - c(n+1)!$$

$$\varphi^{(n+1)}(\tau_t) = f^{(n+1)}(\tau_t) - c(n+1)! = 0$$

□

Thm. 6.2.2.15 $\Rightarrow$

↑

max on both sides

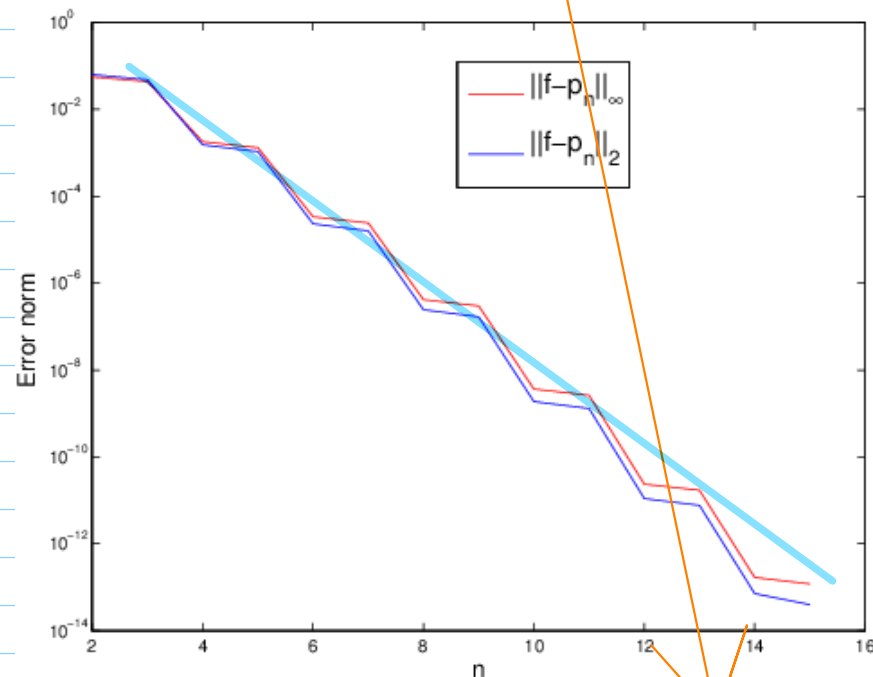
$$\|f - L_T f\|_{L^\infty(I)} \leq \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \max_{t \in I} |(t - t_0) \cdots (t - t_n)|. \quad (6.2.2.21)$$

Quantitative interpolation error estimates rely on smoothness!

Ex 6.2.2.23 ($\rightarrow$ Exp. 6.2.2.4)

$$f(t) := \sin t, \quad I = [0, \pi], \quad \mathcal{I}_n = \left\{ \frac{\pi}{n} j \right\}_{j=0}^n$$

$$\hookrightarrow \|f^{(k)}\|_\infty \leq 1 \quad \forall k$$



"Super"
Exponential
convergence

$$\|f^{(k)}\|_{L^\infty(I)} \leq 1, \quad \forall k \in \mathbb{N}_0 \Rightarrow \|f - p\|_{L^\infty(I)} \leq \frac{1}{(1+n)!} \max_{t \in I} |(t-0)(t-\frac{\pi}{n})(t-\frac{2\pi}{n}) \cdots (t-\pi)|$$

$$\leq \frac{1}{n+1} \left(\frac{\pi}{n}\right)^{n+1} \cdot \max_{t \in I} |t - \frac{\pi}{2n}|$$

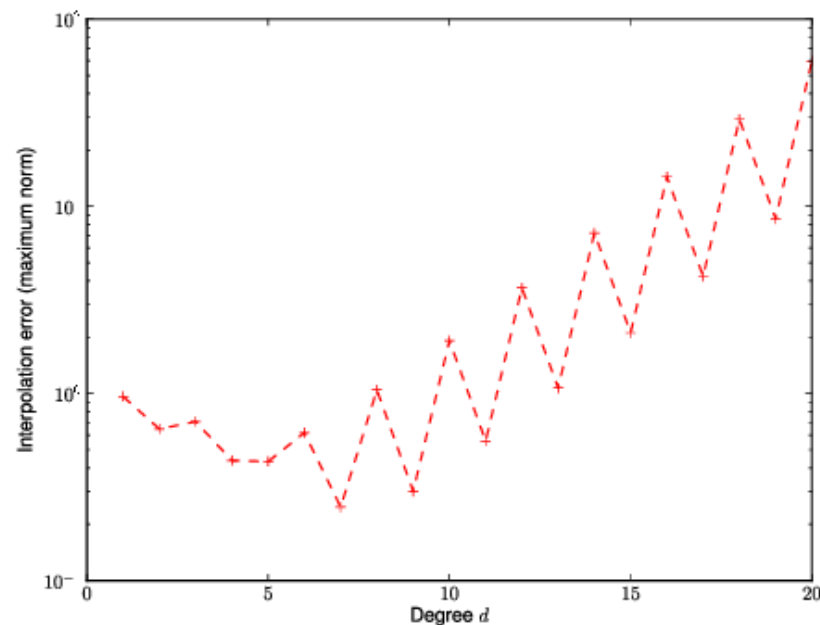
$$\hookrightarrow \rightarrow 0 \text{ for } n \rightarrow \infty$$

(4)

Ex. 6.2.2.24 (Runge's example, Exp. 6.2.2.11)

$$f(t) = \frac{1}{1+t^2}, I = [-5, 5], \mathcal{J}_n = \left\{ -5 + \frac{10}{n}j \right\}_{j=0}^n$$

$\uparrow$ $|f^{(k)}(t)| = 2^k k! O(|t|^{-2-k})$ for $k \rightarrow \infty$



Approximate $\|f - L_{\mathcal{T}_n} f\|_\infty$ on $[-5, 5]$

Blow-up of bound
(6.2.2.21)

§ 6.2.2.28 (Interpolation error estimates and Lebesgue constants $\lambda_{\mathcal{J}}$)

node set

$$\lambda_{\mathcal{T}} := \|\mathcal{I}_{\mathcal{T}}\|_{\infty \rightarrow \infty} := \sup_{\mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\|\mathcal{I}_{\mathcal{T}}(\mathbf{y})\|_{L^\infty(I)}}{\|\mathbf{y}\|_\infty},$$

! key observation: $L_{\mathcal{J}}$ preserves polynomials

$$L_{\mathcal{T}} p = \mathcal{I}_{\mathcal{T}}([p(t)]_{t \in \mathcal{T}}) = p \quad \forall p \in \mathcal{P}_n. \quad n := \#\mathcal{J} - 1 \quad (6.2.2.30)$$

$$\begin{aligned} \|f - L_{\mathcal{J}} f\|_\infty &= \|(f - p) - (L_{\mathcal{J}} f - L_{\mathcal{J}} p)\|_\infty \\ &\leq \|(Id - L_{\mathcal{J}})(f - p)\|_\infty \\ &\leq (1 + \lambda_{\mathcal{J}}) \|f - p\|_\infty \quad \forall p \in \mathcal{P}_n \end{aligned}$$

$$\|L_{\mathcal{J}} g\|_\infty \leq \lambda_{\mathcal{J}} \max_{t \in \mathcal{J}} |g(t)| \leq \lambda_{\mathcal{J}} \|g\|_\infty$$

$$\triangleright \underbrace{\|f - L_{\mathcal{J}} f\|_\infty}_{\text{interpolation error}} \leq (1 + \lambda_{\mathcal{J}}) \underbrace{\inf_{p \in \mathcal{P}_n} \|f - p\|_\infty}_{\text{best approx error}}$$

