

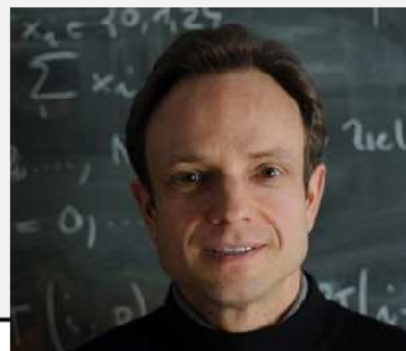
Course Video

Section 6.2.2.3: Error Estimates for Polynomial Interpolation:
Analytic Interpolands

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- **Complex analysis**: path integrals
- **Complex analysis**: residue theorem

Dependencies. Section 6.2.2.1

Duration: minutes

VI. Approximation of Functions
in One Dimension6.2. Approximation by Global
Polynomials6.2.2. Error Estimates for Polynomial
Interpolation

Ex. 6.2.2.23 / Exp. 6.2.2.4

Lagrange polynomial approximation of
 $f(t) = \sin t$ on $[0, \pi]$ with equidistant nodes

▷ (Super-) exponential convergence

$$\|f - L_{\mathcal{J}_n} f\|_{\infty} \leq \frac{1}{n+1} \left(\frac{\pi}{n}\right)^{n+1} \quad \forall n \in \mathbb{N}$$

$\in C^{\infty}(\mathbb{R})$

②

6.2.2.3. Analytic Interpolands

$$\sin t = \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} t^{2k+1}}_{\substack{\text{power series} \\ = \text{"polynomial of degree } \infty"}}, \quad \forall t \in \mathbb{R}$$

Definition 6.2.2.35. Real-analytic functions

A function $f \in C^\infty(I)$ defined on an open interval $I \subset \mathbb{R}$ is called **(real-)analytic** on I , if it possesses a convergent Taylor series at every point $t_0 \in I$:

$$\forall t_0 \in I: \exists \rho = \rho(t_0) > 0: f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t - t_0)^n \quad \forall t \in I: |t - t_0| < \rho(t_0).$$

$\rho(t_0)$ is called the **radius of convergence** of the Taylor series.

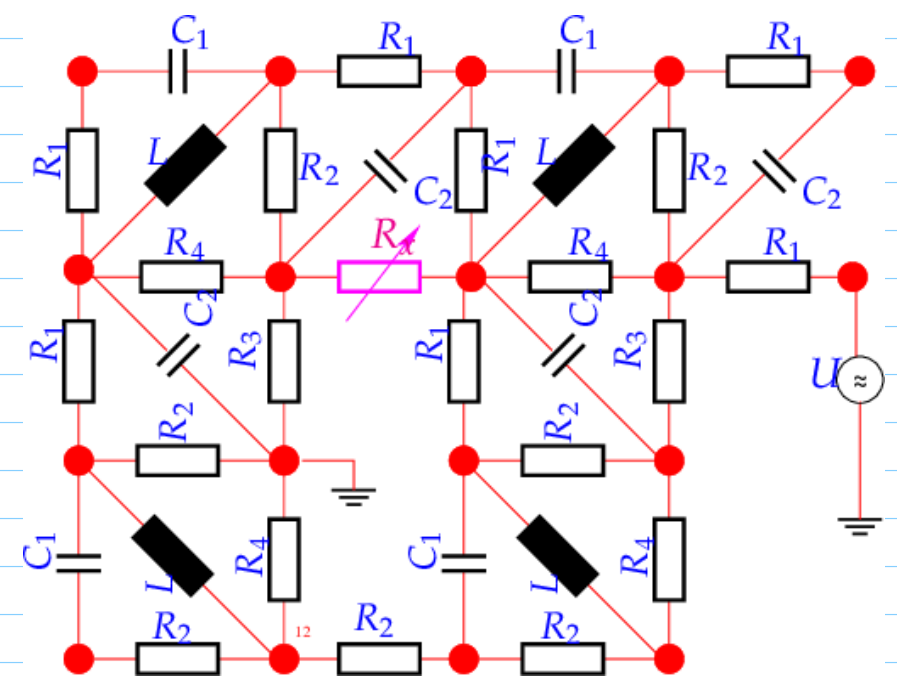
local property

Ex 6.2.2.37 (Hump function)

$$f(t) = \frac{1}{1+t^2} = \sum_{k=0}^{\infty} (-1)^k t^{2k} \quad \text{for } |t| \leq 1$$

$\hat{=}$ a local power series representation

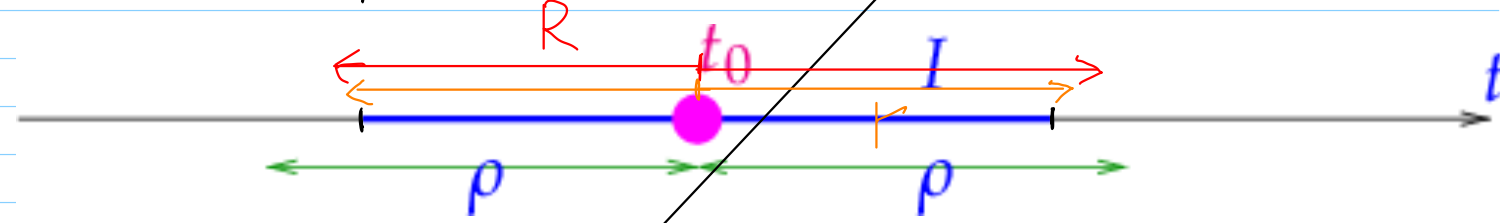
Rem 6.2.2.40 (Analytic functions everywhere)



△ Linear RLC circuit
All currents (voltages) are analytic functions of R_x !

§ 6.2.2.41 (Approximation by truncated power series)

$$f(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k \quad \forall |t - t_0| < \rho$$



$$p_n(t) := \sum_{k=0}^n a_k (t - t_0)^k \in \mathcal{P}_n. \quad (6.2.2.43)$$

③ $r < R < \rho \triangleq \text{radius of convergence}$

$$\sum_{k=0}^{\infty} |a_k| R^k =: C < \infty$$

$\Delta\text{-inequ}$

$$|f(t) - p_n(t)| = \left| \sum_{k=n+1}^{\infty} a_k (t - t_0)^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| R^k \left(\frac{t - t_0}{R} \right)^k \leq C \left(\frac{r}{R} \right)^{n+1} \quad \forall t \in I$$

► $\|f - p_n\|_{L^\infty(I)} \leq C q^{n+1} \quad \forall n \in \mathbb{N} \quad \text{with} \quad q := r/R < 1 \quad (6.2.2.46)$

$\triangleq$ exponential convergence

Required: globally conv. power series

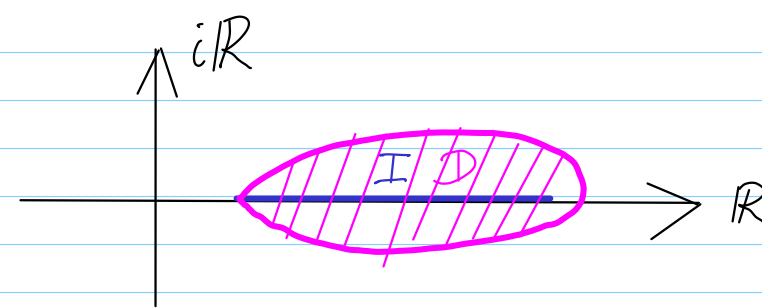
Tool: Complex analysis

$$f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \quad a_n \in \mathbb{R}, \quad \forall t \in \mathbb{R}: |t - t_0| < \rho(t_0),$$

► $f(z) = \sum_{n=0}^{\infty} a_n (z - t_0)^n, \quad \forall z \in \mathbb{C}: |z - t_0| < \rho(t_0).$

Real- and complex-analytic functions

Every real-analytic function on $I \subset \mathbb{R}$ can be extended to a complex-analytic function on some open set $D \subset \mathbb{C}$ with $I \subset D$.



Definition 6.2.2.48. Analyticity of a complex valued function

Let $D \subset \mathbb{C}$ be an open set in the complex plane. A function $f : D \rightarrow \mathbb{C}$ is called (complex)-analytic/holomorphic in D , for every point $z \in D$ one can find $\rho(z) > 0$ and $a_k \in \mathbb{C}$, $k \in \mathbb{N}_0$, such that

$$f(w) = \sum_{k=0}^{\infty} a_k (w - z)^k \quad \forall w \in D: |z - w| < \rho(z).$$

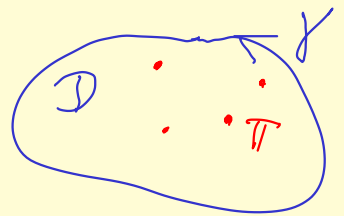
Theorem 6.2.2.50. Residue theorem

Let $D \subset \mathbb{C}$ be an open set, $G \subset D$ a closed set contained in D , $\gamma := \partial G$ its (piecewise smooth and oriented) boundary, and Π a finite set contained in the interior of G .

Then for each function f that is analytic in $\bar{D} \setminus \Pi$ holds

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{p \in \Pi} \text{res}_p f,$$

where $\text{res}_p f$ is the residual of f in $p \in \mathbb{C}$.



complex path integral $\in \mathbb{C}$

$$\left| \int_{\gamma} f(z) dz \right| \leq \underbrace{|\gamma|}_{\text{length of } \gamma} \cdot \max_{z \in \gamma} |f(z)|$$

Lemma 6.2.2.53. Residual formula for quotients

let g and h be complex valued functions that are both analytic in a neighborhood of $p \in \mathbb{C}$, and satisfy $h(p) = 0$, $h'(p) \neq 0$. Then

$$\operatorname{res}_p \frac{g}{h} = \frac{g(p)}{h'(p)}.$$

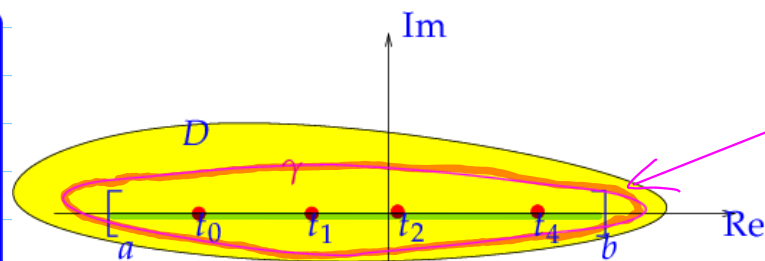
↑ has a pole in p

§ 6.2.2.54 (Residue remainder formula for Lagrange interpolation)

$I = [a, b]$, node set $\mathcal{J} = \{t_0, \dots, t_n\} \subset I$

Assumption 6.2.2.55. Analyticity of interpoland

We assume that the interpoland $f: I \rightarrow \mathbb{C}$ can be extended to a function $f: D \subset \mathbb{C} \rightarrow \mathbb{C}$, which is **analytic** ($\rightarrow$ Def. 6.2.2.48) on the open set $D \subset \mathbb{C}$ with $[a, b] \subset D$.



$\hookrightarrow [a, b]$ has a positive distance from ∂D

$$L_{\mathcal{J}} f(t) = \sum_{j=0}^n f(t_j) L_j(t)$$

Nodal polynomial $w(t) = \prod_{k=0}^n (t - t_k)$

$$w'(t_j) = \prod_{\substack{k=0 \\ k \neq j}}^n (t_j - t_k) \quad [\text{product rule}]$$

$$L_j(t) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{(t - t_k)}{(t_j - t_k)} = \frac{w(t)}{(t - t_j) w'(t_j)}$$

Auxiliary function, fix $t \in I \setminus \mathcal{J}$

$$g_t(z) = \frac{f(z)}{(z - t)w(z)} \quad z \in D \setminus \Pi$$

$$\Pi := \{t_0, \dots, t_n, t\}$$

Analytic on $D \setminus \Pi$, denominator has only single zeros

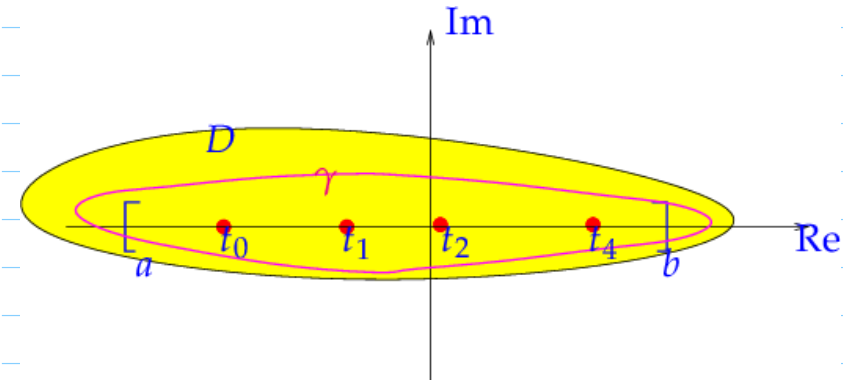
$$\frac{1}{2\pi i} \oint_{\gamma} g_t(z) dz \stackrel{\text{res. Thm}}{=} \operatorname{res}_t g_t + \sum_{j=0}^n \operatorname{res}_{t_j} g_t$$

$$= \frac{f(t)}{w(t)} + \sum_{j=0}^n \frac{f(t_j)}{(t_j - t)w'(t_j)} \quad \{\cdot w\}$$

$$f(t) = \underbrace{\sum_{j=1}^n f(t_j) \frac{w(t)}{(t_j - t)w'(t_j)}}_{\text{polynomial interpolant!}} + \underbrace{\frac{w(t)}{2\pi i} \int_{\gamma} g_t(z) dz}_{\text{interpolation error!}}. \quad (6.2.2.57)$$

5

$$|f(t) - L_T f(t)| = \left| \frac{w(t)}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-t)w(z)} dz \right| \leq \frac{|\gamma| \max_{a \leq \tau \leq b} |w(\tau)|}{2\pi \min_{z \in \gamma} |w(z)|} \cdot \frac{\max_{z \in \gamma} |f(z)|}{\text{dist}([a, b], \gamma)}. \quad (6.2.2.58)$$



For studying vrg of
intp. error we need
only existence of
 D and γ .

§ 6.2.2.59 (Determining the domain of analyticity)

Ex: $f(t) = \frac{1}{1+t^2} = \frac{1}{(1+it)(1-it)}$

▷ Analytic in $\mathbb{C} \setminus \{\pm i\}$

Theorem 6.2.2.60. Composition and products of analytic functions

If $f, h : D \subset \mathbb{C} \rightarrow \mathbb{C}$ and $g : U \subset \mathbb{C} \rightarrow \mathbb{C}$ are analytic in the open sets D and U , respectively, then

- (i) the composition $f \circ g$ is analytic in $\{z \in U : g(z) \in D\}$,
- (ii) the product $f \cdot h$ is analytic on D .

Ex: $t \mapsto \sqrt{1+t^2}$ analytic in $\mathbb{C} \setminus \pm [1, \infty]i$

Note: $z \mapsto \sqrt{z}$ is analytic in $\mathbb{C} \setminus]-\infty, 0]$
 $1+t^2 = -\xi, \xi \geq 0$

