

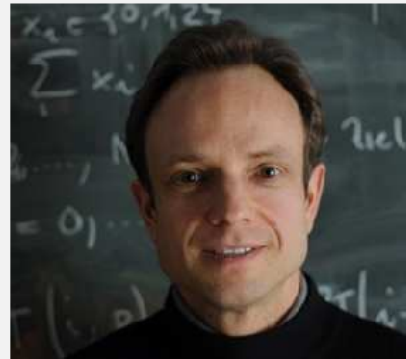
Course Video

Section 6.2.2: Error Estimates for Polynomial Interpolation

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Date: November 9, 2020

(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Calculus in one variable
- Complex analysis: theory of holomorphic functions

Dependencies. Section 5.2.1

Duration: minutes

VI. Approximation of Functions in One Dimension

6.2. Approximation by Global Polynomials

6.2.2. Error Estimates for Polynomial Interpolation

Lagrange interpolation

Interpolation scheme + sampling → approximation scheme

$$f : I \subset \mathbb{R} \rightarrow \mathbb{K} \xrightarrow{\text{sampling}} (t_i, y_i := f(t_i))_{i=0}^m \xrightarrow{\text{interpolation}} \tilde{f} := \mathcal{I}_{\mathcal{T}} \mathbf{y} \quad (\tilde{f}(t_i) = y_i).$$

↑
free choice of nodes $t_i \in I$

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Definition 6.2.2.1. Lagrangian (interpolation polynomial) approximation scheme

Given an interval $I \subset \mathbb{R}$, $n \in \mathbb{N}$, a node set $\mathcal{T} = \{t_0, \dots, t_n\} \subset I$, the Lagrangian (interpolation polynomial) approximation scheme $L_{\mathcal{T}} : C^0(I) \rightarrow \mathcal{P}_n$ is defined by

$$L_{\mathcal{T}}(f) := l_{\mathcal{T}}(\mathbf{y}) \in \mathcal{P}_n \quad \text{with} \quad \mathbf{y} := (f(t_0), \dots, f(t_n))^T \in \mathbb{K}^{n+1},$$

$$l_{\mathcal{T}}(\mathbf{y})(t_j) = (\mathbf{y})_j, \quad j = 0, \dots, n.$$

$f \longmapsto$ interpolating polynomial

6.2.2.1 Convergence of Interpolation Errors

$\hookrightarrow$ Consider sequences of node sets

$$(\mathcal{T}_n)_{n \in \mathbb{N}}, \quad \#\mathcal{T} = n+1 \Leftrightarrow \text{pol. deg.} \rightarrow \infty$$

Ex.: $I = [a, b]$: equidistant nodes

$$\mathcal{T}_n := \{t_j^{(n)} := a + (b-a)\frac{j}{n} : j = 0, \dots, n\} \subset I. \quad (6.2.2.3)$$

$\triangleright$ Find "sharp" bound $T : \mathbb{N} \rightarrow \mathbb{R}^+$

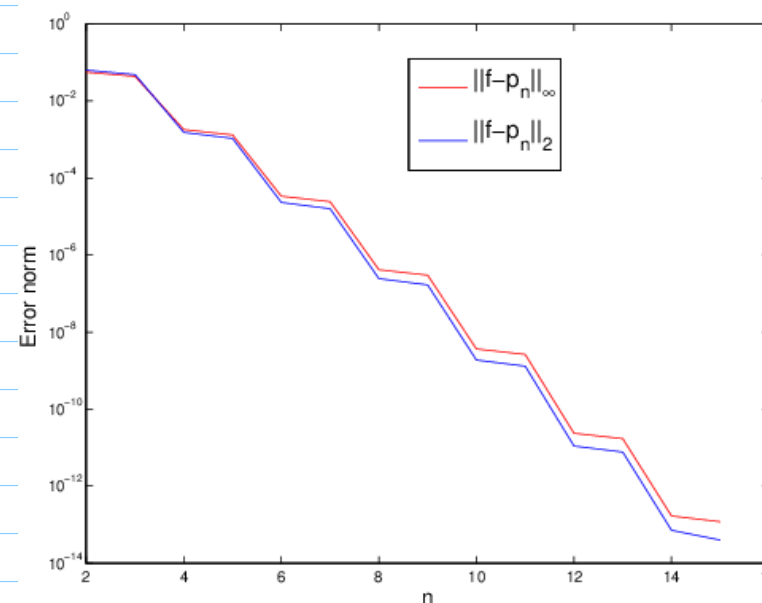
$$\exists C \neq C(n) > 0: \|f - L_{\mathcal{T}_n} f\| \leq C T(n) \quad \text{for } n \rightarrow \infty. \quad (6.2.2.6)$$

$\uparrow$
independent of n !

$\hookrightarrow$ "generic constant", generally unknown

Exp. 6.2.2.4 : (Asymptotic behavior of Lagrange interpolation error)

$$f(t) := \sin t, \quad I = [0, \pi], \quad \mathcal{T}_n = \left\{ \frac{\pi}{n} j \right\}_{j=0}^n$$



$\triangle \|f - L_{\mathcal{T}_n} f\|_{\infty}$ and $\|f - L_{\mathcal{T}_n} f\|_2$
 $\uparrow$
seems to be governed by a law

§ 6.2.2.5 (Classification of asymptotic behavior of the interpolation error)

Definition 6.2.2.7. Types of asymptotic convergence of approximation schemes

Writing $T(n)$ for the bound of the norm of the interpolation error according to (6.2.2.6) we distinguish the following types of asymptotic behavior:

$\exists p > 0: T(n) \leq n^{-p}$: algebraic convergence, with rate $p > 0$, $\forall n \in \mathbb{N}$.

$\exists 0 < q < 1: T(n) \leq q^n$: exponential convergence, $\|f - p_n\| = O(q^n)$ for $n \rightarrow \infty$

The bounds are assumed to be sharp in the sense, that no bounds with larger rate p (for algebraic convergence) or smaller q (for exponential convergence) can be found.

③

§ 6.2.2.9 (Determining type of convergence in numerical experiments)

"Measured": $(n_i, \epsilon_i) \in \mathcal{N} \times \mathbb{R}^+$ for $i=1,2,\dots$
 $\uparrow$ error norm

❶ Conjectured: **algebraic convergence**: $\epsilon_i \approx C n_i^{-p}$, $C > 0$ unknown

$$\log(\epsilon_i) \approx \log(C) - p \log n_i \quad (\text{affine linear in log-log scale})$$

$$\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right) \approx -p \log\left(\frac{n_i}{n_{i-1}}\right) \longrightarrow p$$

$n \nearrow ?$ for $\epsilon \longrightarrow \epsilon/2$



❶ Conjectured: **exponential convergence**: $\epsilon_i \approx C \exp(-\beta n_i)$

$$\log \epsilon_i \approx \log(C) - \beta n_i \quad (\text{affine linear in lin-log scale})$$

$$n \nearrow ? \quad \text{for} \quad \epsilon \longrightarrow \epsilon/2$$



