

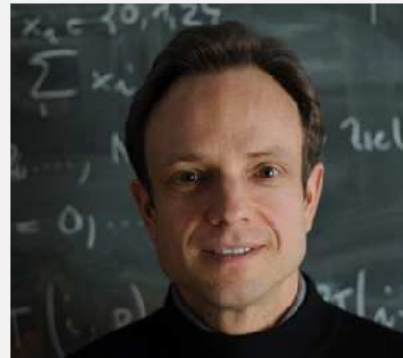
Course Video

Section 6.2.3.1: Chebychev Interpolation:
Motivation and Definition

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Dependencies. Section 6.2.2.2

Duration: 2/ minutes

VI. Approximation of Functions
in One Dimension6.2. Approximation by Global
Polynomials

6.2.3. Chebychev Interpolation

Interpolation scheme + sampling $\rightarrow$ approximation scheme

$$f : I \subset \mathbb{R} \rightarrow \mathbb{K} \xrightarrow{\text{sampling}} (t_i, y_i := f(t_i))_{i=0}^m \xrightarrow{\text{interpolation}} \tilde{f} := l_{\mathcal{T}} \mathbf{y} \quad (\tilde{f}(t_i) = y_i).$$

↑
free choice of nodes $t_i \in I$

"Bad choice" : equidistant nodes, Exp. 6.2.2.11

6.2.3.1 Motivation and Definition

Goal: "A priori optimal nodes"
 $\rightarrow$ independent of f

Thm. 6.2.2.15 $\Rightarrow$

$$\|f - \mathcal{L}_T f\|_{L^\infty(I)} \leq \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \underbrace{\max_{t \in I} |(t - t_0) \cdots (t - t_n)|}_w. \quad (6.2.2.22)$$

$= \|w\|_{\infty, I}, w(t) = \prod_{j=0}^n (t - t_j)$
 $\rightarrow$ make as small as possible

Optimal choice of interpolation nodes independent of interpoland



Idea: choose nodes $t_0, \dots, t_n$ such that $\|w\|_{L^\infty(I)}$ is minimal!

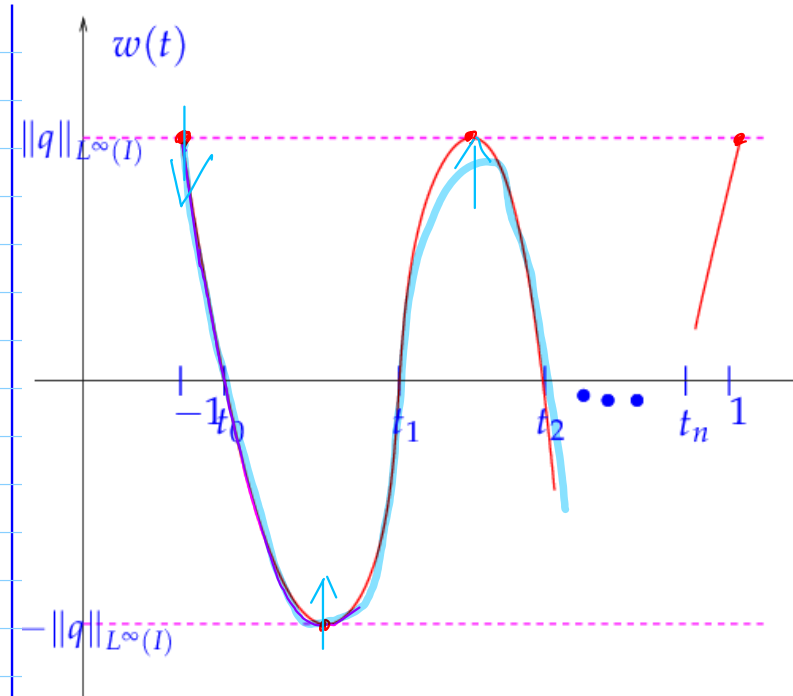
This is equivalent to finding a polynomial $q \in \mathcal{P}_{n+1}$

- ♦ with leading coefficient = 1,
- ♦ such that it minimizes the norm $\|q\|_{L^\infty(I)}$.



Then choose nodes $t_0, \dots, t_n$ as **zeros** of q .
 (caution: all t_j must lie in I !)

First $I = [-1, 1]$



Necessary shape requirements:

$n+2$ equal- $| \cdot |$ extrema with alternating signs

Definition 6.2.3.3. Chebyshev polynomials $\rightarrow$

The n^{th} Chebyshev polynomial is $T_n(t) := \cos(n \arccos t)$, $-1 \leq t \leq 1, n \in \mathbb{N}$.

$$\triangleright \|T_n\|_{\infty, [-1, 1]} = 1$$

Theorem 6.2.3.4. 3-term recursion for Chebyshev polynomials $\rightarrow$

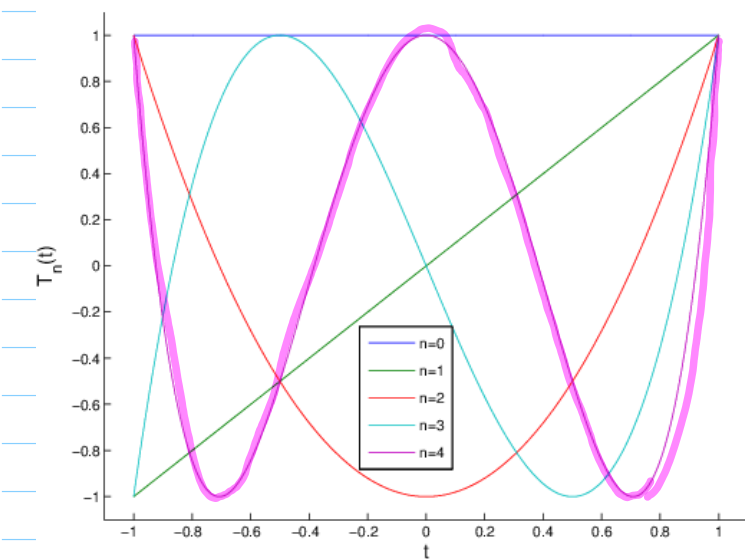
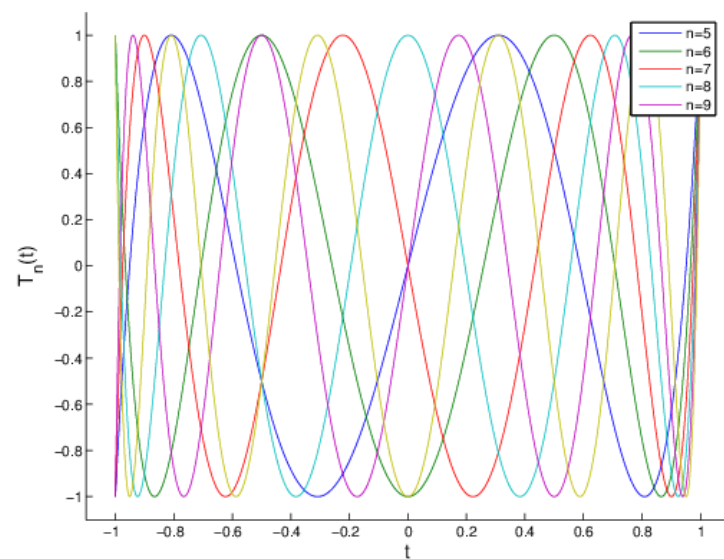
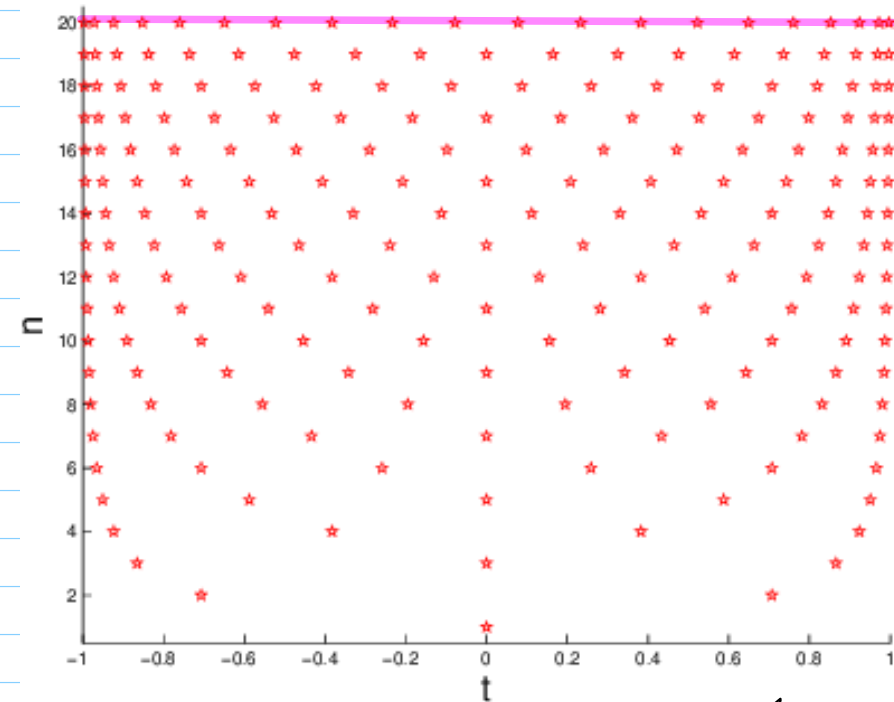
The function T_n defined in Def. 6.2.3.3 satisfy the **3-term recursion**

$$T_{n+1}(t) = 2t T_n(t) - T_{n-1}(t), \quad T_0 \equiv 1, \quad T_1(t) = t, \quad n \in \mathbb{N}. \quad (6.2.3.5)$$

Proof: $\cos((n+1)x) = 2 \cos nx \cdot \cos x - \cos((n-1)x)$

$$\triangleright T_n \in \mathcal{P}_n, \text{ leading coefficient } 2^{n-1}$$

③

Chebyshev polynomials $T_0, \dots, T_4$ Chebyshev polynomials $T_5, \dots, T_9$ 

◁ Cheb nodes cluster at ± 1

On $[a, b]$: $t_j = \phi(\hat{t}_j) = a + \frac{1}{2}(b-a)(1 + \hat{t}_j)$

Theorem 6.2.3.8. Minimax property of the Chebyshev polynomials

The polynomials T_n from Def. 6.2.3.3 minimize the supremum norm in the following sense:

$$\|T_n\|_{L^\infty([-1,1])} = \inf\{\|p\|_{L^\infty([-1,1])} : p \in \mathcal{P}_n, p(t) = 2^{n-1}t^n + \dots\}, \quad \forall n \in \mathbb{N}.$$

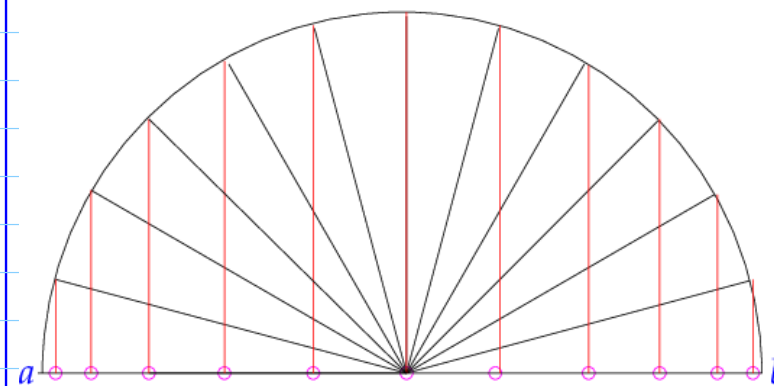
↑ leading coefficient

▷ A priori optimal interpolation nodes $\hat{t}_j$: zeros of T_{n+1}

$$\hat{t}_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \quad j=0, \dots, n$$

↑ Chebyshev nodes in $[-1, 1]$

Nodal polynomial: $w(t) = 2^{-n} T_{n+1}(t)$



The Chebyshev nodes in the interval $I = [a, b]$ are

$$t_k := a + \frac{1}{2}(b-a) \left(\cos\left(\frac{2k+1}{2(n+1)}\pi\right) + 1 \right), \quad (6.2.3.13)$$

$k = 0, \dots, n$.

↓ Chebyshev interpolation

