

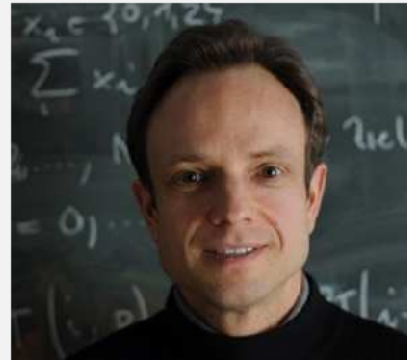
Course Video

Section 6.2.3.2: Chebychev Interpolation Error Estimates

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Date: November 13, 2020

(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependencies. Section 6.2.3.1, Section 6.2.2.3

Duration: 26 minutes

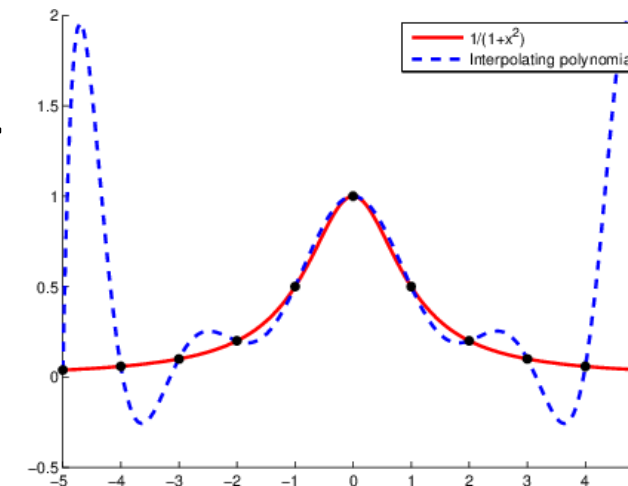
VI. Approximation of Functions in One Dimension

6.2. Approximation by Global Polynomials

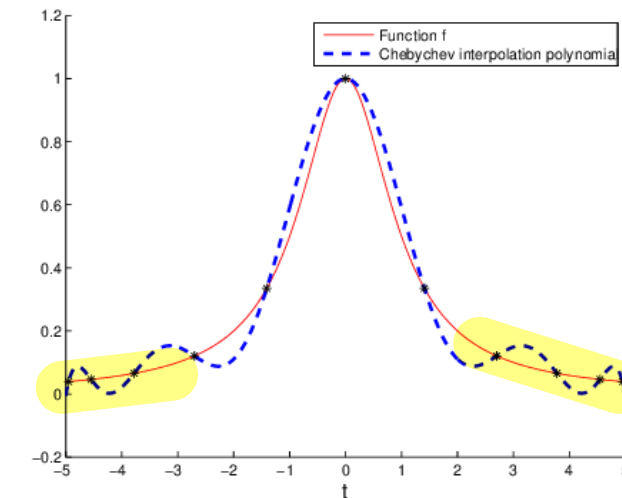
6.2.3. Chebychev Interpolation

Ex. 6.2.3.15 (Chebychev vs. equidistant polynomial intp.)

$$f(t) = \frac{1}{1+t^2}$$



Equidistant nodes



Chebychev nodes

6.2.3.2. Chebychev Interpolation Error Estimates

§ 6.2.3.14 : (Finite-smoothness error estimates for Chebychev interpolation)

Recall Thm. 6.2.2.15:

$$I = [-1, 1]$$

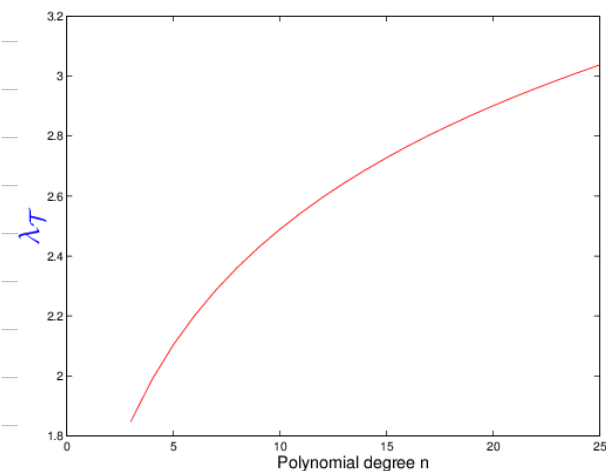
$$\|f - L_{\mathcal{T}}f\|_{L^\infty(I)} \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{L^\infty(I)} \underbrace{\|w\|_{L^\infty(I)}}_{2^{-n}},$$

with nodal polynomial $w(t) := (t - t_0) \cdots (t - t_n)$
 $= 2^{-n} T_{n+1}(t)$

Affine hrf to $[a, b]$:

$$\begin{aligned} \|f - L_{\mathcal{T}}(f)\|_{L^\infty(I)} &= \|\hat{f} - L_{\hat{\mathcal{T}}}(\hat{f})\|_{L^\infty([-1,1])} \leq \frac{2^{-n}}{(n+1)!} \left\| \frac{d^{n+1}\hat{f}}{d\hat{t}^{n+1}} \right\|_{L^\infty([-1,1])} \\ &\leq \frac{2^{-2n-1}}{(n+1)!} |I|^{n+1} \|f^{(n+1)}\|_{L^\infty(I)}. \end{aligned} \quad (6.2.3.17)$$

Rem. 6.2.3.18 (Lebesgue constant for Chebychev nodes)



△ "Measured"

$$\lambda_{\mathcal{T}} \leq \frac{2}{\pi} \log(n+1)$$

[Equid. nodes: $\lambda_{\mathcal{T}} \geq C e^{n/2}$]

$$\|f - L_{\mathcal{T}}f\|_{L^\infty(I)} \leq (1 + \lambda_{\mathcal{T}}) \inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty(I)} \quad \forall f \in C^0(I), \quad (6.2.2.33)$$

Theorem 6.2.1.11. L^∞ polynomial best approximation estimate

If $f \in C^r([-1, 1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then, for any polynomial degree $n \geq r$,

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} \leq (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \|f^{(r)}\|_{L^\infty([-1,1])},$$

where

$$\|f^{(r)}\|_{L^\infty([-1,1])} := \max_{x \in [-1,1]} |f^{(r)}(x)|.$$

$$\triangleright \|f - L_{\mathcal{T}}f\|_{L^\infty([-1,1])} \leq (2/\pi \log(1+n) + 2)(1 + \pi^2/2)^r \frac{(n-r)!}{n!} \|f^{(r)}\|_{L^\infty([-1,1])}. \quad (6.2.3.20)$$

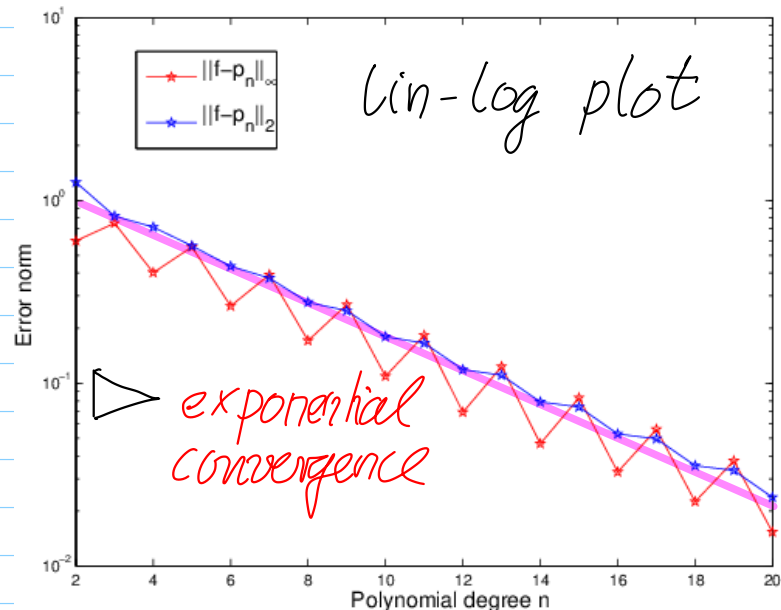
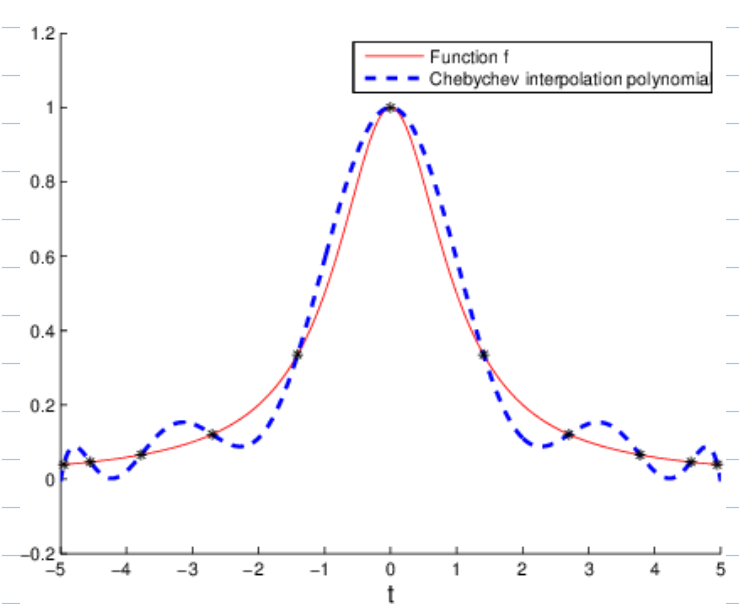
$$= O(\log^n / n^r)$$

("almost algebraic convergence")

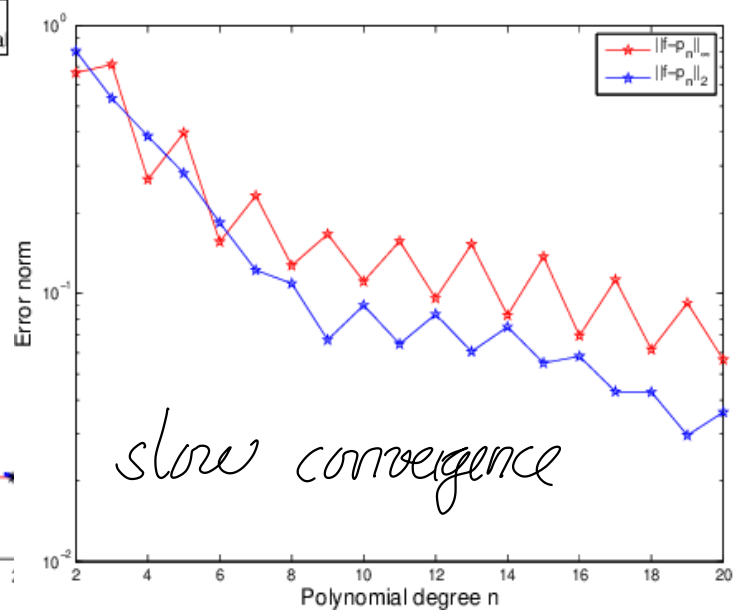
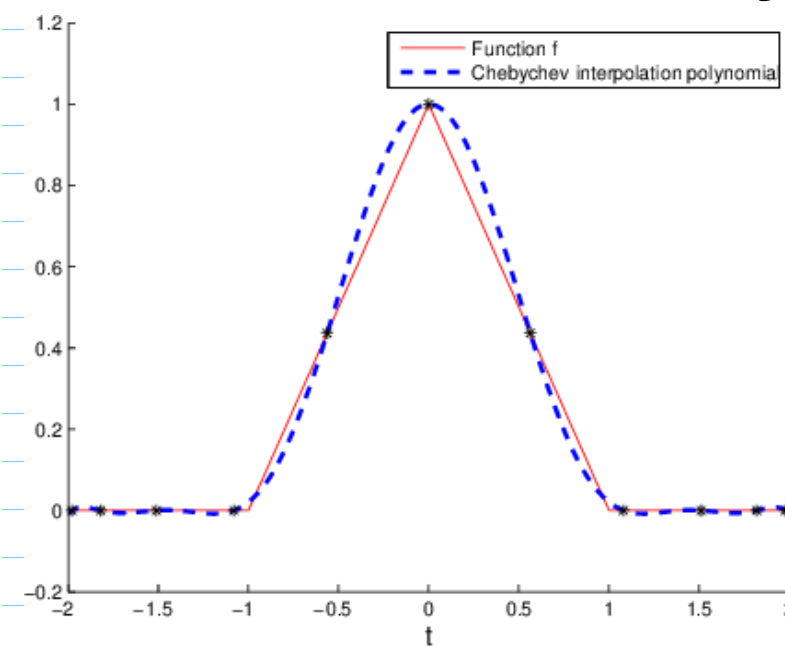
→ If only $f \in C^1([-1,1]) \Rightarrow$ conv. guaranteed

Exp. 6.2.3.21 (Chebychev interpolation error)

① $f(t) = \frac{1}{1+t^2}$ on $[-5, 5]$



② $f(t) = \max\{1 - |t|, 0\}$, $f \in C^0$, $f \notin C^1$



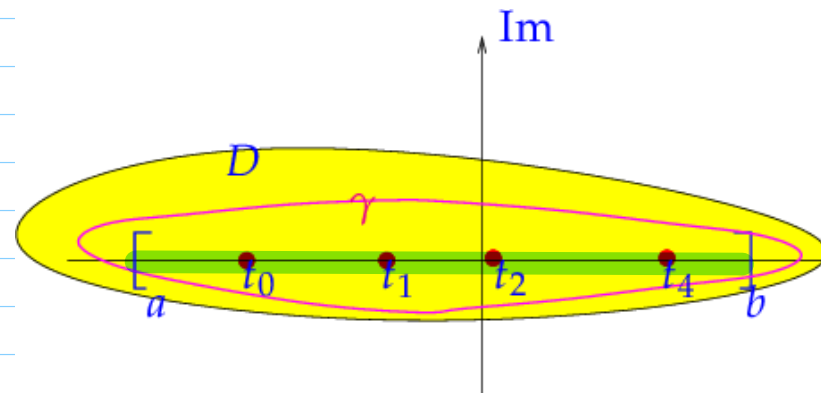
▷ Smoothness of f crucial

Rem. 6.2.3.24 : (Chebychev interpolation of analytic functions)

$I = [-1, 1]$,

$w = 2^{-n} T_{n+1} : \leq 2^{-n}$

$$|f(t) - L_T f(t)| \leq \left| \frac{w(t)}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-t)w(z)} dz \right| \leq \frac{|\gamma| \cdot \max_{|z| \leq 1} |w(\tau)|}{2\pi \min_{z \in \gamma} |w(z)|} \cdot \frac{\max_{z \in \gamma} |f(z)|}{\text{dist}([a, b], \gamma)}, \quad (6.2.2.58)$$

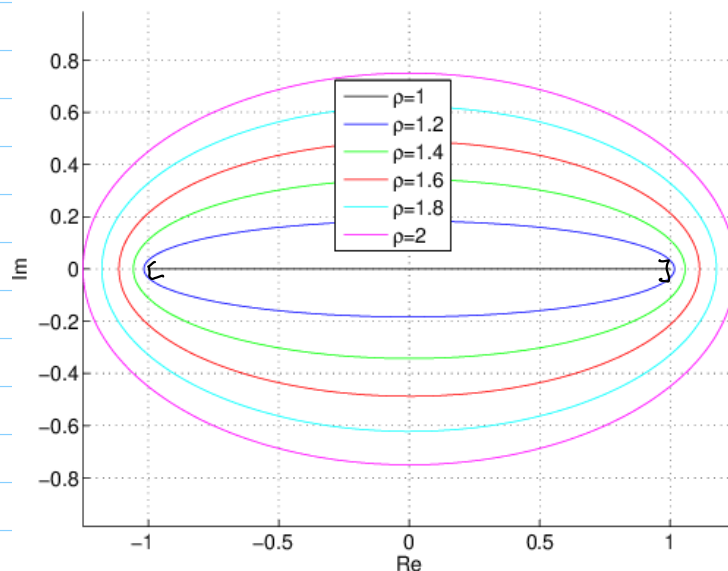


convenience choice of ρ : $(\rho > 1)$

$y(\theta) = \cos(\theta - i \log \rho)$
 $0 \leq \theta \leq 2\pi$

to cancel arccos

4 ellipses around $[-1, 1]$



(4)

$$\begin{aligned}
 |2^n w(\gamma(\theta))|^2 &= |T_{n+1}(\cos(\theta - i \log \rho))|^2 && \text{arccos} \circ \cos = \text{id}! \\
 &= |\cos((n+1)(\theta - i \log \rho))|^2 \\
 &= \cos((n+1)(\theta - i \log \rho)) \cdot \overline{\cos((n+1)(\theta - i \log \rho))} \\
 &= \frac{1}{4}(\rho^s e^{i(n+1)\theta} + \rho^{-(n+1)} e^{-i(n+1)\theta})(\rho^{(n+1)} e^{-i(n+1)\theta} + \rho^{-(n+1)} e^{i(n+1)\theta}) \\
 &= \frac{1}{4}(\rho^{2(n+1)} + \rho^{-2(n+1)} + e^{2i(n+1)\theta} + e^{-2i(n+1)\theta}) \\
 &= \frac{1}{4}(\rho^{n+1} - \underbrace{\rho^{-(n+1)}}_{<1})^2 + \underbrace{\frac{1}{4}(e^{i(n+1)\theta} + e^{-i(n+1)\theta})^2}_{=\cos^2((n+1)\theta) \geq 0} \geq \frac{1}{4}(\rho^{n+1} - 1)^2,
 \end{aligned}$$

$$\|f - L_{\mathcal{T}} f\|_{L^\infty([-1,1])} \leq \underbrace{\frac{2|\gamma|}{\pi} \frac{1}{(\rho^{n+1} - 1)(\rho + \rho^{-1} - 2)}}_{\Rightarrow O(1/\rho^n) \text{ as } n \rightarrow \infty} \cdot \max_{z \in \gamma} |f(z)|. \quad (6.2.3.26)$$

$\Rightarrow O(1/\rho^n)$ as $n \rightarrow \infty$

$\stackrel{1}{=} \text{exponential cvg.}$

for any f w/ analytic extension beyond $[-1,1]$

Slower for smaller domains $D \subset \mathbb{C}$ of analyticity

