

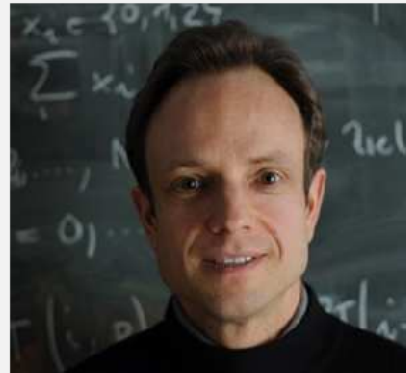
Course Video

Section 6.5.2: Trigonometric Interpolation Error Estimates

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Fundamentals of Fourier analysis

Dependencies. Section 4.2.6, Section 6.5.1

Duration: 26 minutes

VI. Approximation of Functions
in 1D

6.5 Approximation by Trigonometric Polynomials

6.5.2. Trigonometric Interpolation Error Estimates

$$T_n : C_{\text{per}}^0([0,1]) \longrightarrow \mathcal{P}_{2n}^T, \\ (T_n f)(t_k) = f(t_k), \quad t_k := \frac{k}{2n+1} \quad n \leq N$$

Want to study $\|f - T_n f\|_*$ for $n \rightarrow \infty$,
depending on *smoothness* of f

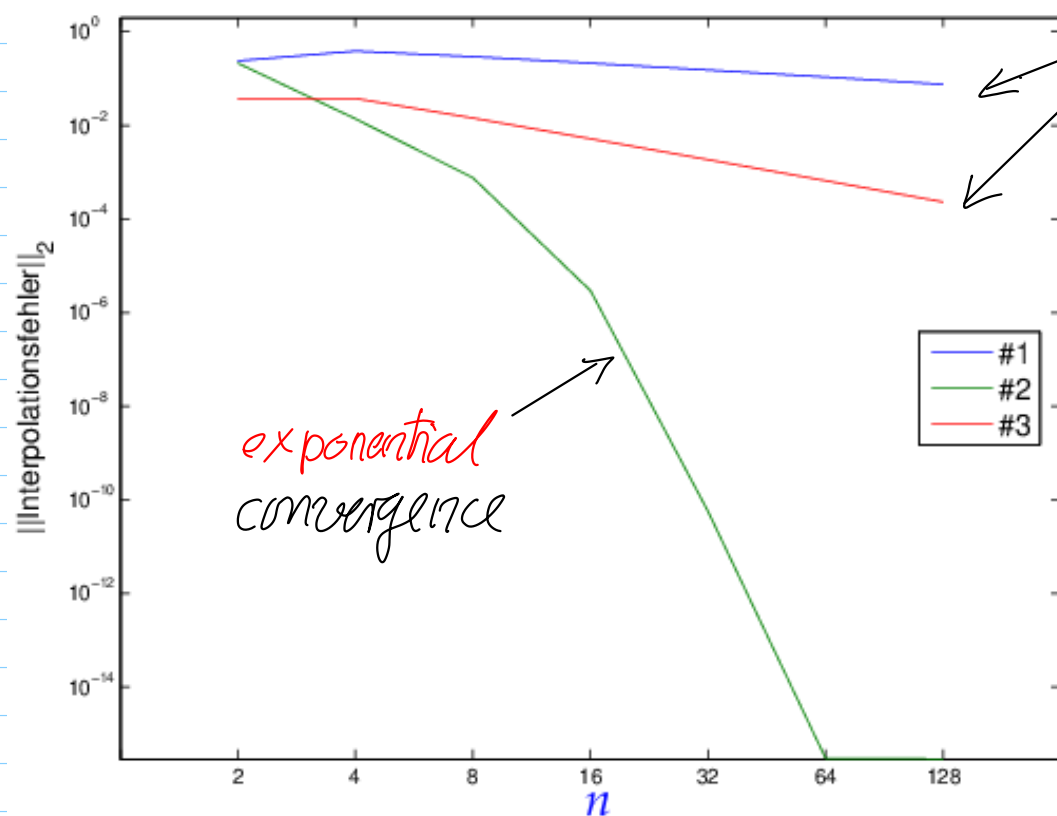
$$* = L^2([0,1]) \quad (L^\infty([0,1]))$$

②

Exp. 6.5.2.1: (Trigonometric interpolation error)

#1 Step function: $f(t) = 0$ for $|t - \frac{1}{2}| > \frac{1}{4}$, $f(t) = 1$ for $|t - \frac{1}{2}| \leq \frac{1}{4} \rightarrow \notin C^0$ #2 C^∞ periodic function: $f(t) = \frac{1}{\sqrt{1 + \frac{1}{2} \sin(2\pi t)}}$ #3 "wedge function": $f(t) = |t - \frac{1}{2}| \rightarrow \notin C^1$

log-log plot

algebraic
convergenceexponential
convergence $L^2([0,1])$ -norm of interpolation error§ 6.5.2.3 (Fourier series $\rightarrow$ 4.2.6)

$$p \in \mathcal{P}_{2n}^T \Rightarrow p(t) = \sum_{k=-n}^n a_k \exp(-2\pi i k t) \text{ for some } a_k \in \mathbb{C}. \quad (6.5.2.4)$$

Fourier series of $f \in L^2([0,1])$

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}_k \exp(-2\pi i k t) \text{ in } L^2([0,1]), \quad \hat{f}_k := \int_0^1 f(t) \exp(2\pi i k t) dt. \quad (6.5.2.5)$$

$\uparrow$ Fourier mode μ_k $\rightarrow$ Fourier coefficient



Idea:

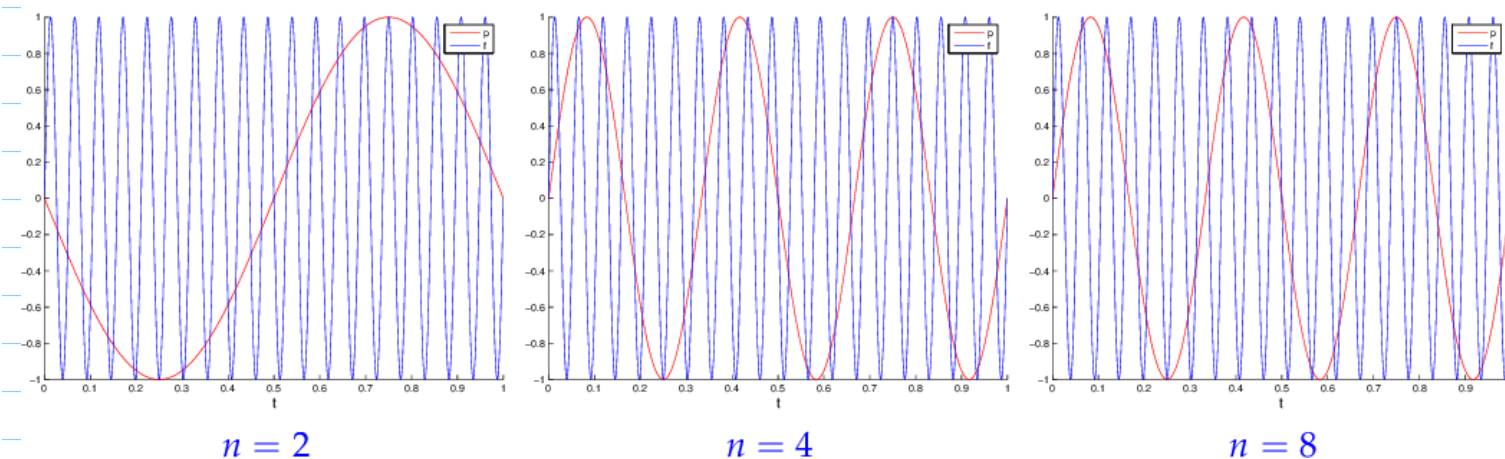
Study trigonometric interpolation error for interpolands given in Fourier series representation (6.5.2.5)

§ 6.5.2.7 (Aliasing)

$$\mathcal{J}_n := \left\{ \frac{k}{2n+1} \right\}_{k=0}^{2n}, \quad \mu_k(t) := e^{-2\pi i k t}$$

$$\mu_k|_{\mathcal{J}_n} = \mu_{k+N}|_{\mathcal{J}_n} \quad \forall N \in (2n+1)\mathbb{Z}$$

3



$$\triangleright T_n \mu_k = T \mu_{\tilde{k}} = \mu_{\tilde{k}} \quad \begin{matrix} -n \leq \tilde{k} \leq n \\ \tilde{k} \equiv k \pmod{2n+1} \end{matrix}$$

$$f(t) = \sum_{j=-\infty}^{\infty} \hat{f}_j \mu_j(t) \quad \blacktriangleright \quad T_n(f)(t) = \sum_{j=-n}^n \gamma_j \mu_j(t), \quad \gamma_j = \sum_{\ell=-\infty}^{\ell} \hat{f}_{j+\ell(2n+1)}. \quad (6.5.2.10)$$

$\triangleright$ Interpolation error

$$E = f - T_n f = \sum_{j \in \mathbb{Z}} \hat{E}_j \mu_j$$

$$\hat{E}_j = \begin{cases} -\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \hat{f}_{j+\ell(2n+1)}, & \text{if } j \in \{-n, \dots, n\}, \\ \hat{f}_j, & \text{if } |j| > n, \end{cases} \quad j \in \mathbb{Z}. \quad (6.5.2.11)$$

$\uparrow$
 $f_j - f_j$

Theorem 4.2.6.33. Isometry property of the Fourier transform

If the Fourier coefficients satisfy $\sum_{k \in \mathbb{Z}} |\hat{c}_k|^2 < \infty$, then the Fourier series

$$c(t) = \sum_{k \in \mathbb{Z}} \hat{c}_k \exp(-2\pi i k t)$$

yields a function $c \in L^2([0,1])$ that satisfies

$$\|c\|_{L^2([0,1])}^2 := \int_0^1 |c(t)|^2 dt = \sum_{k \in \mathbb{Z}} |\hat{c}_k|^2. \quad \rightarrow \quad \hat{c}_j \text{ bounded}$$

$$\|f - T_n f\|_{L^2([0,1])}^2 = \sum_{j=-n}^n \left| \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \hat{f}_{j+\ell(2n+1)} \right|^2 + \sum_{|j| > n} |\hat{f}_j|^2, \quad (6.5.2.13)$$

$$\left[\|f - T_n f\|_{L^\infty([0,1])} \leq \sum_{j=-n}^n \sum_{\ell \in \mathbb{Z} \setminus \{0\}} |\hat{f}_{j+\ell(2n+1)}| + \sum_{|j| > n} |\hat{f}_j|. \right] \quad (6.5.2.14)$$

$\uparrow$ by Δ -inequality

Decay laws for $\hat{f}_j \rightarrow$ estimates

Smoothness of f

④

$$f(t) = \sum_{j \in \mathbb{Z}} \hat{f}_j e^{-2\pi i j t}$$

$$\frac{df}{dt}(t) = \sum_{j \in \mathbb{Z}} \underbrace{\hat{f}_j (-2\pi i j)}_{= (\frac{d\hat{f}}{dt})_j} e^{-2\pi i j t}$$

Lemma 6.5.2.17. Fourier coefficients of derivatives

For the Fourier coefficients of the derivatives a 1-periodic function $f \in C^{k-1}(\mathbb{R})$, $k \in \mathbb{N}$, with integrable k -th derivative $f^{(k)}$ holds

$$\widehat{(f^{(k)})}_j = (-2\pi i j)^k \hat{f}_j, \quad j \in \mathbb{Z}.$$

$\hookrightarrow$ bounded

Lemma 6.5.2.20. Decay of Fourier coefficients

If $f \in C^{k-1}(\mathbb{R})$ with integrable k -th derivative, then $\hat{f}_j = O(|j|^{-k})$ for $|j| \rightarrow \infty$

Theorem 6.5.2.23. Finite-smoothness L^2 -error estimate for trigonometric interpolation

If $f \in C^{k-1}(\mathbb{R})$, $k \in \mathbb{N}$, with square-integrable k -th derivative ($f^{(k)} \in L^2([0,1])$), then

$$\|f - T_n f\|_{L^2([0,1])} \leq \sqrt{1 + c_k} (2\pi n)^{-k} \|f^{(k)}\|_{L^2([0,1])}, \quad (6.5.2.24)$$

with $c_k = 2 \sum_{\ell=1}^{\infty} (2\ell - 1)^{-2k} < \infty$.

↑
Algebraic convergence, rate k

