

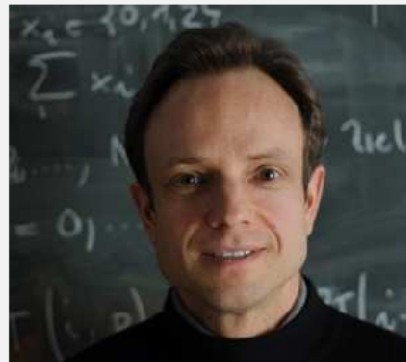
Course Video

Section 6.5.3: Trigonometric Interpolation of Analytic Periodic Functions

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Elementary complex analysis

Dependencies. Section 6.5.1, Section 6.2.2.3

Duration: 25 minutes

VI. Approximation of Functions in 1D

6.5 Approximation by Trigonometric Polynomials

6.5.3. Trigonometric Interpolation of Analytic Periodic Functions

Definition 6.2.2.35. Real-analytic functions

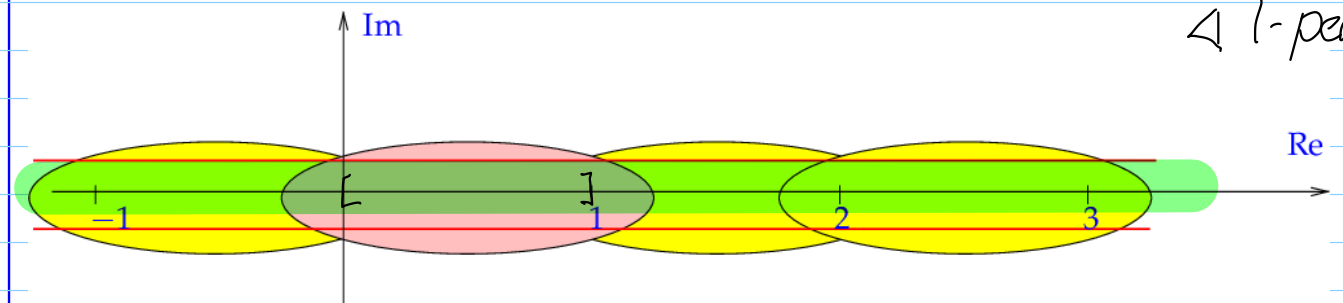
A function $f \in C^\infty(I)$ defined on an open interval $I \subset \mathbb{R}$ is called **(real-)analytic** on I , if it possesses a convergent Taylor series at every point $t_0 \in I$:

$$\forall t_0 \in I: \exists \rho = \rho(t_0) > 0: f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!} (z - t_0)^k \quad \forall z \in I: |z - t_0| < \rho(t_0).$$

$\rho(t_0)$ is called the **radius of convergence** of the Taylor series.

Naturally defined on $\mathbb{D} \subset \mathbb{C}$

1-periodic

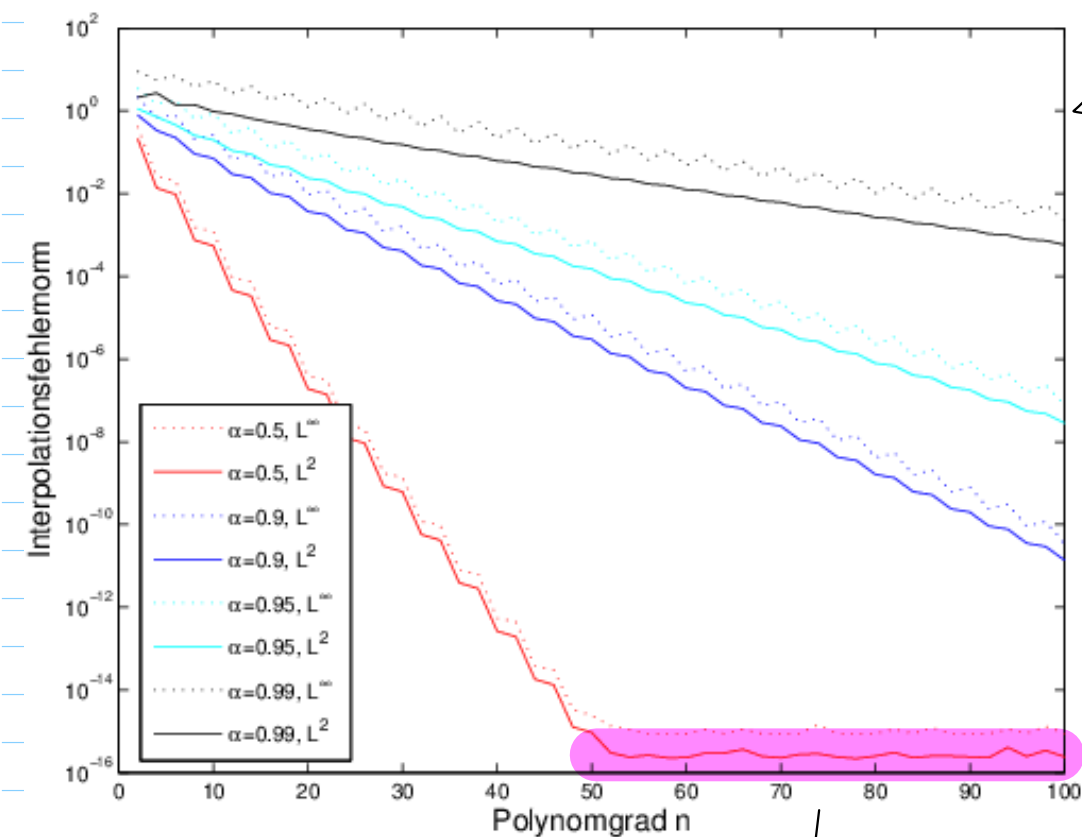


②

Exp. 6.5.3.1 (Trigonometric Interpolation of analytic functions)

$$f(t) = \frac{1}{\sqrt{1 - \alpha \sin(2\pi t)}} \quad , \quad t \in \mathbb{R}$$

[equidistant] $0 < \alpha < 1$



↓
?

§ 6.5.3.3 (Decay of Fourier coefficients of analytic functions)

Theorem 6.5.3.4. Exponential decay of Fourier coefficients of analytic functions

If $f : \mathbb{R} \rightarrow \mathbb{C}$ is 1-periodic and has an **analytic extension** to the strip

$$\bar{S} := \{z \in \mathbb{C} : -\eta \leq \text{Im} z \leq \eta\} \quad , \quad \text{for some } \eta > 0 \quad ,$$

then its Fourier coefficients decay according to

$$|\hat{f}_k| \leq q^{|k|} \cdot \|f\|_{L^\infty(\bar{S})} \quad \forall k \in \mathbb{Z} \quad \text{with} \quad q := \exp(-2\pi\eta) \in]0, 1[\quad . \quad (6.5.3.5)$$

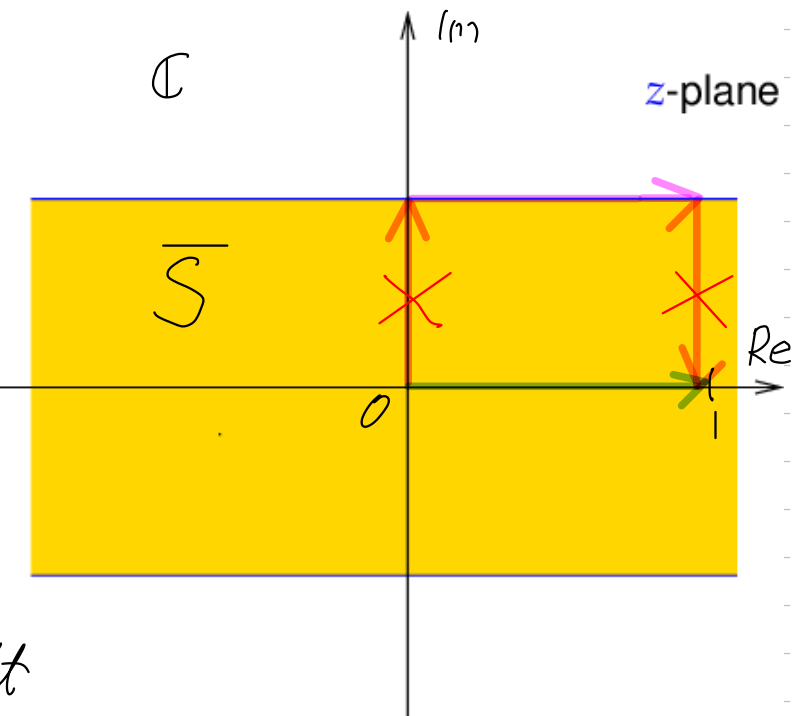
Proof: $k \geq 0$

$$\hat{f}_k = \int_0^1 f(t) e^{2\pi i k t} dt$$

by **Cauchy integral theorem**

$$\begin{aligned} \hat{f}_k &= \int_0^1 f(t + i\eta) e^{2\pi i k (t + i\eta)} dt \\ &= e^{-2\pi k \eta} \int_0^1 f(t + i\eta) e^{2\pi i k t} dt \end{aligned}$$

$$|\hat{f}_k| \leq e^{-2\pi k \eta} \|f\|_{\infty, \bar{S}}$$



③ Recall :

$$\|f - T_n f\|_{L^2([0,1])}^2 = \sum_{j=-n}^n \left| \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \hat{f}_{j+\ell(2n+1)} \right|^2 + \sum_{|j|>n} |\hat{f}_j|^2, \quad (6.5.2.13)$$

$$\|f - T_n f\|_{L^\infty([0,1])} \leq \sum_{j=-n}^n \sum_{\ell \in \mathbb{Z} \setminus \{0\}} |\hat{f}_{j+\ell(2n+1)}| + \sum_{|j|>n} |\hat{f}_j|. \quad (6.5.2.14)$$

↑
equid. trig. intp into $\mathbb{P}_{2n}^T$

Theorem 6.5.3.13. Exponential convergence of trigonometric interpolation for analytic interpolands

If $f : \mathbb{R} \rightarrow \mathbb{C}$ is 1-periodic and possesses an *analytic extension* to the strip

$$\bar{S} := \{z \in \mathbb{C} : -\eta \leq \operatorname{Im} z \leq \eta\}, \quad \text{for some } \eta > 0,$$

then there is $C_\eta > 0$ depending only on η such that

$$\|f - T_n f\|_* \leq C_\eta e^{-\pi\eta n} \|f\|_{L^\infty(\bar{S})}, \quad n \in \mathbb{N}, \quad (* = L^2([0,1]), L^\infty([0,1])). \quad (6.5.3.14)$$

↑
exponential convergence

§ 6.5.3.15 (Exp. 6.5.3.1. explained)

$$f(t) = \frac{1}{\sqrt{1 - \alpha \sin(2\pi t)}}, \quad t \in \mathbb{R}, \quad 0 < \alpha < 1$$

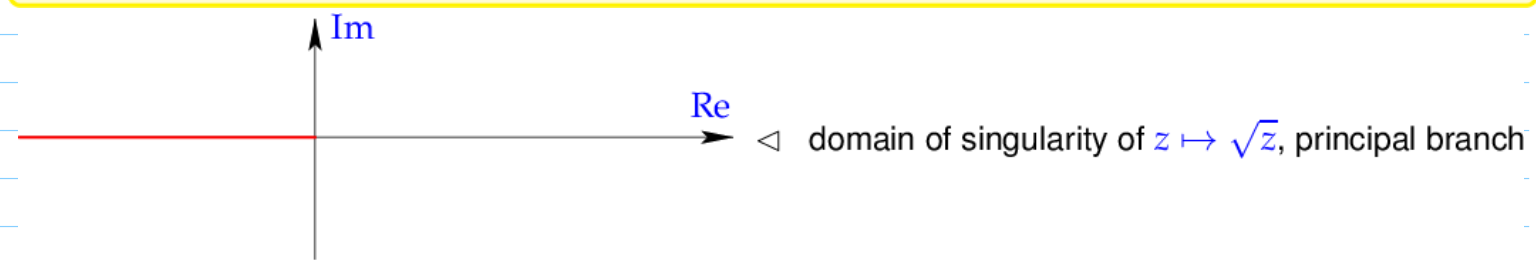
Goal: Find domain of analyticity $D \subset \mathbb{C}$ of f

Easier: "Domain singularity" $P := \mathbb{C} \setminus D$

Lemma 6.5.3.17. Principal branch of the square root

The square root function $t \mapsto \sqrt{t}$, $t \geq 0$, can be extended to an analytic function on

$$\mathbb{C} \setminus \mathbb{R}_0^- := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ or } \operatorname{Im}(z) \neq 0\}.$$



$$P := \{z \in \mathbb{C} : 1 - \alpha \sin(2\pi z) \in \mathbb{R}_0^-\}$$

$$z = x + iy:$$

$$1 + \alpha \sin(2\pi z) \in \mathbb{R}_0^-$$

⇔

$$\sin(2\pi z) = \sin(2\pi x) \cosh(2\pi y) + i \cos(2\pi x) \sinh(2\pi y) \in]-\infty, -1 - \frac{1}{\alpha}]$$

⇔

$$\sin(2\pi x) \cosh(2\pi y) \leq -1 - \frac{1}{\alpha} \quad \text{and} \quad \cos(2\pi x) \sinh(2\pi y) = 0.$$

(4)

 $\Rightarrow$

$$y \neq 0 \quad \forall \quad x \in \mathbb{Z} + \frac{1}{2}$$

 $\hookrightarrow$ contradicts

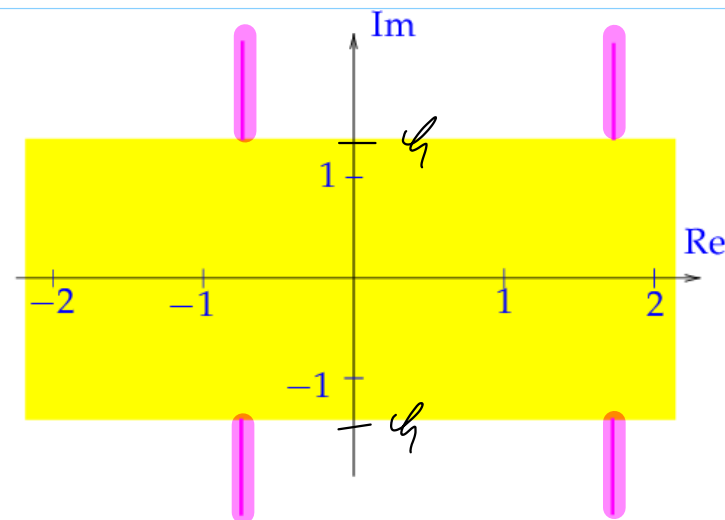
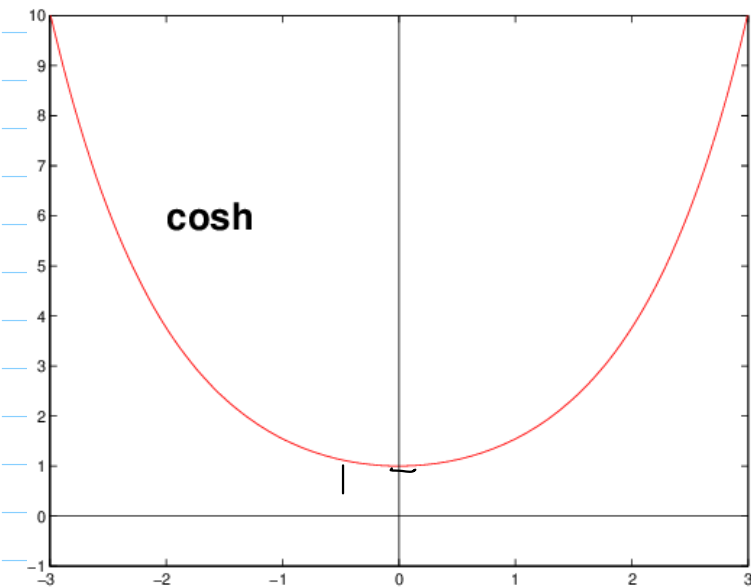
$$\sin(2\pi x) \in \{1, -1\}$$

 $\Downarrow$

$$x \in \mathbb{Z} + \frac{3}{4}$$

 $\Rightarrow$

$$\cosh(2\pi y) \geq 1 + \frac{1}{2}$$


 $\triangleright$ f analytic in strip

$$S := \{z \in \mathbb{C} : -\zeta < \text{Im}(z) < \zeta\}.$$

$$\cosh(2\pi \zeta) = 1 + \frac{1}{2}$$

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