

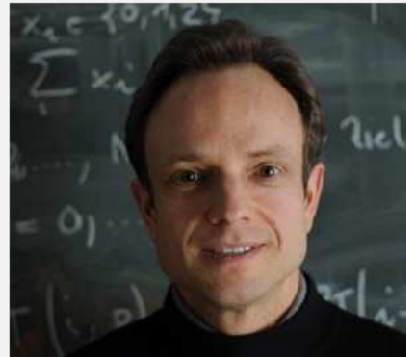
Course Video

Section 6.6.1: Piecewise Polynomial Lagrange Interpolation

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Dependencies. Section 5.2.2, Section 5.3.2, Section 6.2.2.2,

Duration: 30 minutes

VI Approximation of Functions in 1D

6.6. Approximation by Piecewise Polynomials

↓
?

Interpolation scheme + sampling → approximation scheme

$$f : I \subset \mathbb{R} \rightarrow \mathbb{K} \xrightarrow{\text{sampling}} (t_i, y_i := f(t_i))_{i=0}^m \xrightarrow{\text{interpolation}} \tilde{f} := l_{\mathcal{T}} \mathbf{y} \quad (\tilde{f}(t_i) = y_i).$$

↑
free choice of nodes $t_i \in I$

Now: p.w. polynomial

(Ch. 5: p.w. linear, cubic)

- Benefits :
- Locality
 - Cheap algorithm
 - shape preservation

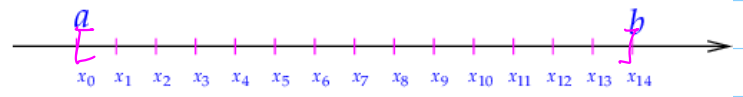
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Idea: use **piecewise polynomials** with respect to a **grid/mesh**
 $\rightarrow$ partition of $I := [a, b]$: *free to choose*
 $\mathcal{M} := \{a = x_0 < x_1 < \dots < x_{m-1} < x_m = b\}$ (6.6.0.2)
 to approximate function $f : [a, b] \mapsto \mathbb{R}, a < b$.

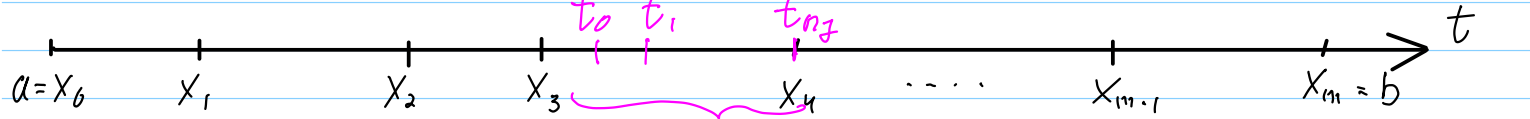
Terminology:

- ◆ $x_j \hat{=}$ **nodes** of the mesh $\mathcal{M}$,
- ◆ $[x_{j-1}, x_j[\hat{=}$ **intervals/cells** of the mesh,
- ◆ $h_{\mathcal{M}} := \max_j |x_j - x_{j-1}| \hat{=}$ **mesh width**,
- ◆ If $x_j = a + jh \hat{=}$ **equidistant** (uniform) mesh with meshwidth $h > 0$



6.6.1 Piecewise Polynomial Lagrange Interpolation

Idea: Local Lagrange polynomial interpolation on cells
 Fix polynomial degree n_j for j -th cell $[x_{j-1}, x_j]$
 $t_0^j, t_1^j, \dots, t_{n_j}^j$
 $n_j + 1$ interpolation nodes $\in [x_{j-1}, x_j]$



General local Lagrange interpolation on a mesh (PPLIP)

- 1 Choose **local degree** $n_j \in \mathbb{N}_0$ for each cell of the mesh, $j = 1, \dots, m$.
- 2 Choose set of **local** interpolation nodes

$$\mathcal{T}^j := \{t_0^j, \dots, t_{n_j}^j\} \subset I_j := [x_{j-1}, x_j], \quad j = 1, \dots, m,$$

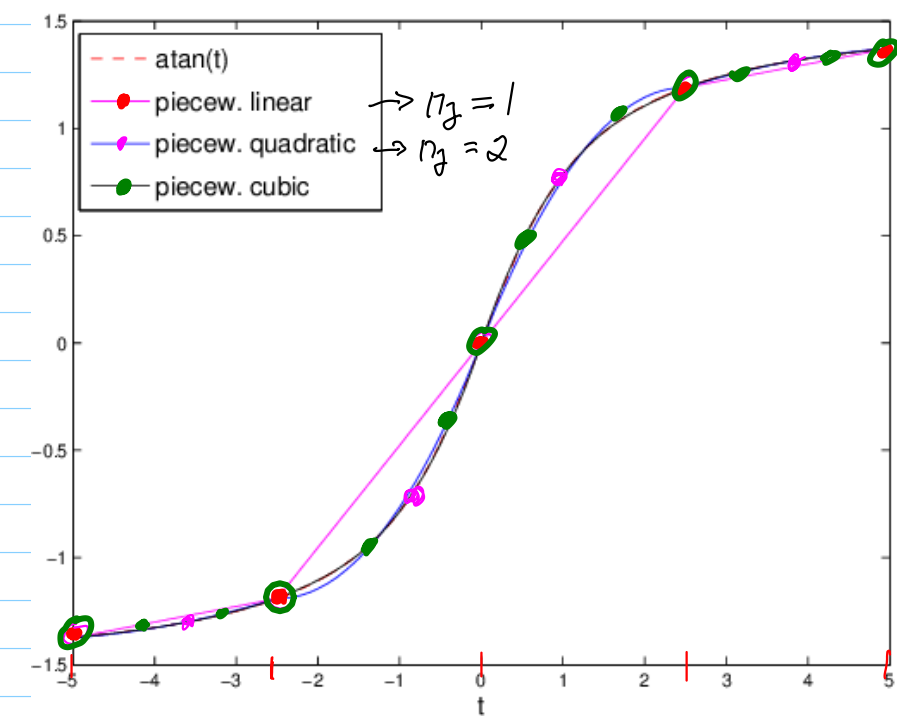
for each mesh cell/grid interval I_j .

- 3 Define **piecewise polynomial** (PP) interpolant $s : [x_0, x_m] \rightarrow \mathbb{K}$:

$$s_j := s|_{I_j} \in \mathcal{P}_{n_j} \quad \text{and} \quad s_j(t_i^j) = f(t_i^j) \quad i = 0, \dots, n_j, \quad j = 1, \dots, m. \quad (6.6.1.2)$$

Owing to Thm. 5.2.2.7, s_j is well defined.

I.C. $\triangleright$ Linear interpolation operator $\mathcal{I}_m$



$$\mathcal{M} = \{-5, -5/2, 0, 5/2, 5\}$$

p.w - quadratic:

$$t_0^j = x_{j-1}$$

$$t_1^j = \frac{1}{2}(x_j + x_{j-1})$$

$$t_2^j = x_j$$

③

PPLIP - interpolant $\in C^0([a,b])$?Yes $\Leftrightarrow t_{n_j}^j = x_j = t_0^{j+1}, j=1, \dots, m-1$  C^1 continuity? $\rightarrow$ not achievable by PPLIP

Focus: Asymptotics of interpolation error

Fix $n_j = n$ & same interpolation scheme $\forall$ cellsConsider sequences of meshes with $h_m \rightarrow 0$ $\hat{=}$ h -convergence

$$\|f - I_m f\|_\infty \leq C T(h_m)$$

 $f \in C^{n+1}(I)$:Thm. 6.2.2.15 $\Rightarrow$

$$\|f - L_T f\|_{L^\infty(I)} \leq \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \max_{t \in I} |(t-t_0) \cdots (t-t_n)|. \quad (6.2.2.22)$$

Apply to $I = [x_{j-1}, x_j]$
 with $n = n_j$

$$\leq h_j^{n+1}, \quad h_j := x_j - x_{j-1}$$

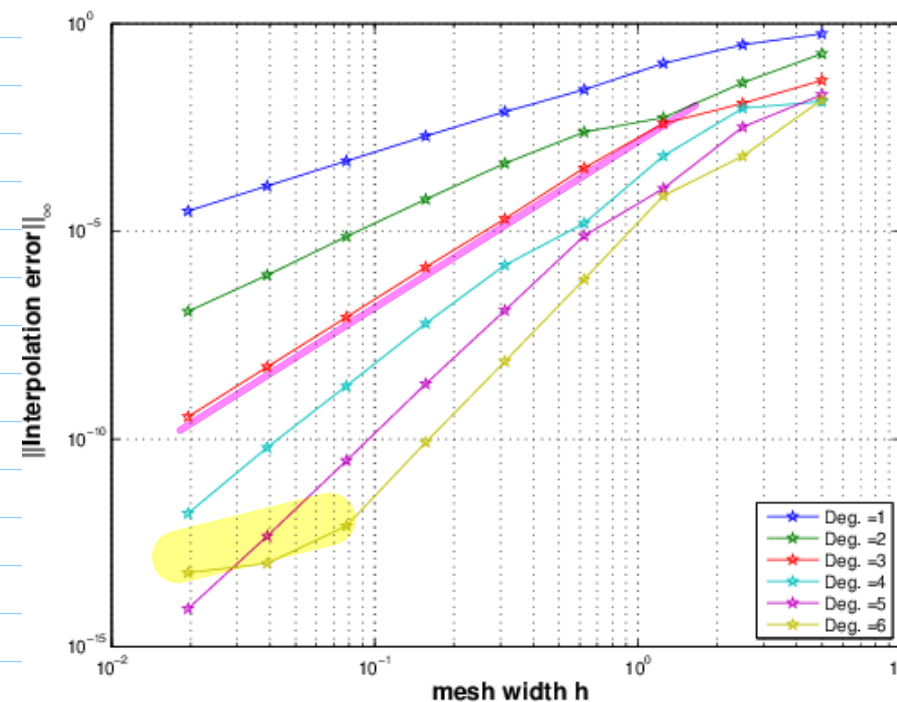
Take max over all cells

$$\|f - I_m f\|_{\infty, [a,b]} \leq \frac{L}{(n+1)!} \|f^{(n+1)}\|_{[a,b], \infty} h_m^{n+1}$$

$$= O(h_m^{n+1})$$

 $\hat{=}$ algebraic ~~exp~~ with rate $n+1$ Ex. 6.6.1.7. (h -convergence of PPLIP) $f(t) = \arctan t$, $I = [-5, 5]$, equid. mesh.

[Equidistant local interpolation nodes]

 $\triangleleft$ log-plot

[error points on lines]

rates $\approx n+1$

