

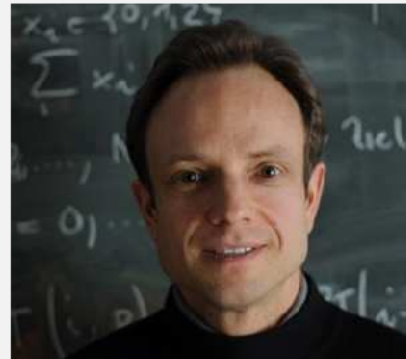
Course Video

Section 6.6.2: Cubic Hermite and Spline Interpolation: Error Estimates

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Dependencies. Section 5.3.3, Section 5.4.2

Duration: minutes

VI Approximation of Functions
in 1D

6.6. Approximation by Piecewise Polynomials

Interpolation scheme + sampling → approximation scheme

$$f: I \subset \mathbb{R} \rightarrow \mathbb{K} \xrightarrow{\text{sampling}} (t_i, y_i := f(t_i))_{i=0}^m \xrightarrow{\text{interpolation}} \tilde{f} := \mathcal{I}_{\mathcal{T}} \mathbf{y} \quad (\tilde{f}(t_i) = y_i) .$$

[generalized]

free choice of nodes $t_i \in I$

cubic Hermite intp.
Sect. 5.5.3

cubic spline intp.
5.4.2

6.6.2. Cubic Hermite Interpolation : Error Estimates

Definition 6.6.2.1. Piecewise cubic Hermite interpolant (with exact slopes) → Def. 5.3.3.1

Given $f \in C^1([a, b])$ and a mesh $\mathcal{M} := \{a = x_0 < x_1 < \dots < x_{m-1} < x_m = b\}$ the **piecewise cubic Hermite interpolant** (with exact slopes) $s : [a, b] \rightarrow \mathbb{R}$ is defined as

$$s|_{[x_{j-1}, x_j]} \in \mathcal{P}_3, \quad j = 1, \dots, m, \quad s(x_j) = f(x_j), \quad s'(x_j) = f'(x_j), \quad j = 0, \dots, m.$$

Generalized interpolation
(also based on derivative values)

$$s \in C^1([a, b])$$

Exp. 6.6.2.2 : (Convergence of Hermite interpolation with exact slopes)

h -convergence :

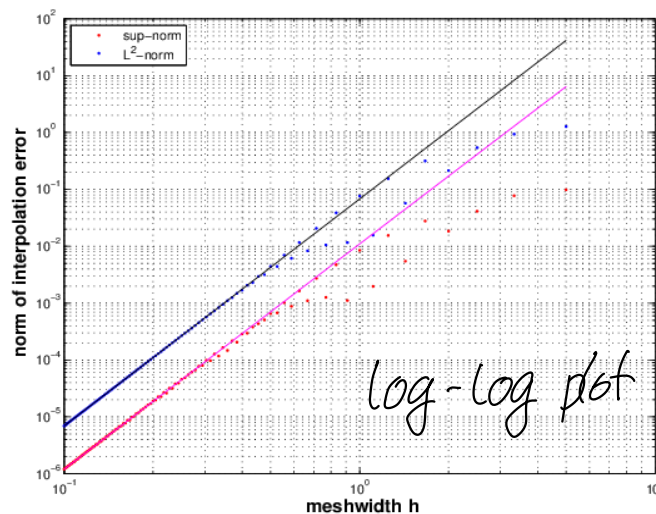
Piecewise cubic Hermite interpolation of

$$f(x) = \arctan(x).$$

- domain: $I = (-5, 5)$
- mesh $\mathcal{T} = \{-5 + hj\}_{j=0}^n \subset I, h = \frac{10}{n}$,
- exact slopes $c_i = f'(t_i), i = 0, \dots, n$

algebraic convergence $O(h^4)$

→ equidistant mesh



Theorem 6.6.2.3. Convergence of approximation by cubic Hermite interpolation

Let s be the cubic Hermite interpolant of $f \in C^4([a, b])$ on a mesh $\mathcal{M} := \{a = x_0 < x_1 < \dots < x_{m-1} < x_m = b\}$ according to Def. 6.6.2.1. Then

$$\|f - s\|_{L^\infty([a, b])} \leq \frac{1}{4!} h_{\mathcal{M}}^4 \|f^{(4)}\|_{L^\infty([a, b])},$$

with the meshwidth $h_{\mathcal{M}} := \max_j |x_j - x_{j-1}|$.

6.6.3. Cubic Spline Interpolation : Error Estimates

- Here: mesh = knot set
- Natural / periodic CSI not meaningful for generic $f : [a, b] \rightarrow \mathbb{R}$

Definition 6.6.3.1. Complete cubic spline interpolant

Given $f \in C^1([a, b])$ and a mesh (= knot set) $\mathcal{M} := \{a = x_0 < x_1 < \dots < x_{m-1} < x_m = b\}$ the **complete cubic spline interpolant** s is defined by the conditions

$$s \in \mathcal{S}_{3, \mathcal{M}}, \quad s(x_j) = f(x_j), \quad j = 0, \dots, m, \quad s'(a) = f'(a), \quad s'(b) = f'(b).$$

→ valid definition

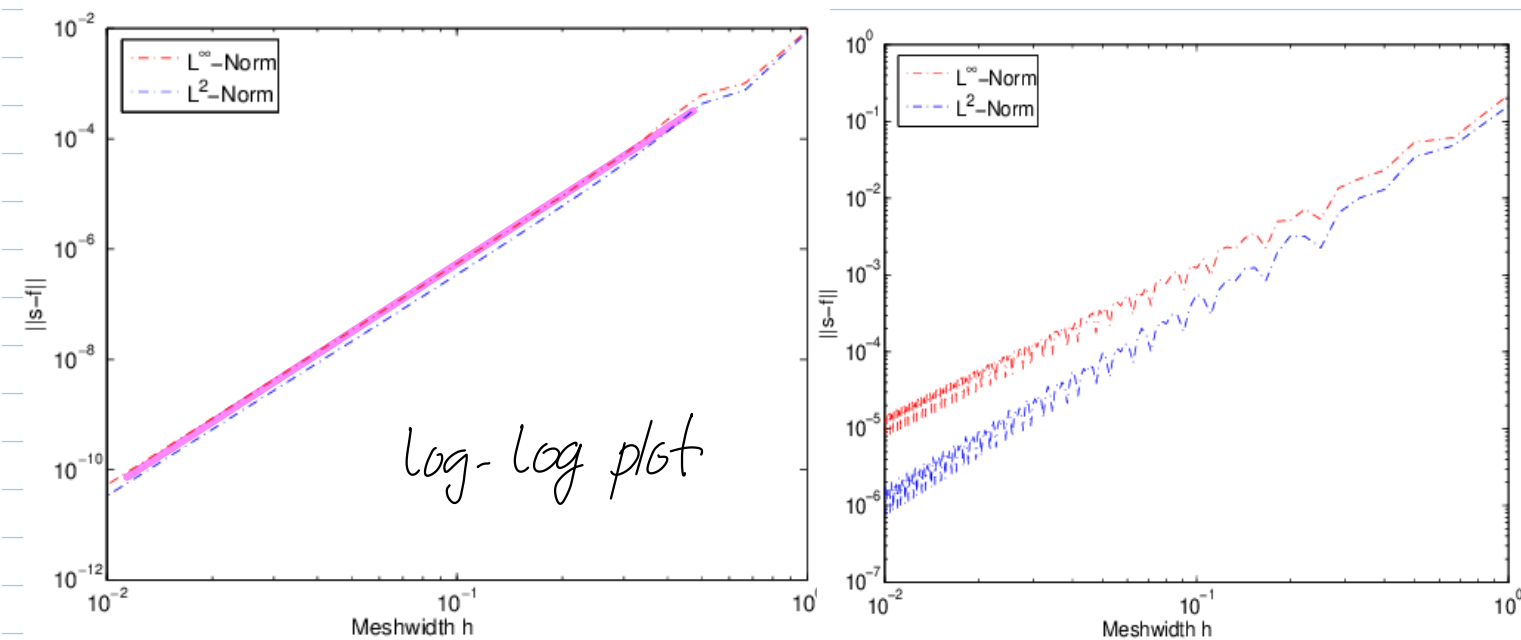
generalized CSI

③

Exp.: 6.6.3.1 (Approximation by complete cubic spline interpolants)

$I = [-1, 1]$, equidistant mesh

$$f_1(t) = \frac{1}{1+e^{-2t}} \in C^\infty(I), \quad f_2(t) = \begin{cases} 0 & , \text{ if } t < -\frac{2}{5}, \\ \frac{1}{2}(1 + \cos(\pi(t - \frac{3}{5}))) & , \text{ if } -\frac{2}{5} < t < \frac{3}{5}, \\ 1 & \text{ otherwise.} \end{cases} \in C^1(I).$$



$$\|f_1 - s\|_{L^\infty([-1,1])} = O(h^4)$$

$$\|f_2 - s\|_{L^\infty([-1,1])} = O(h^2)$$

→ low smoothness limits convergence

Theory :

$$f \in C^4([t_0, t_n]) \quad \|f - s\|_{L^\infty([t_0, t_n])} \leq \frac{5}{384} h^4 \|f^{(4)}\|_{L^\infty([t_0, t_n])}. \quad (6.6.3.3)$$

