

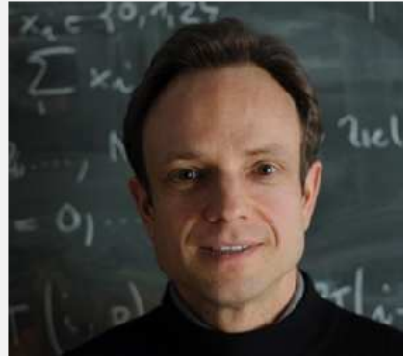
Course Video

Section 7.4.1: Order of a Quadrature Rule

Prof. R. Hiptmair, SAM, ETH Zurich

Date: November 21, 2020

(C) Seminar für Angewandte Mathematik, ETH Zürich

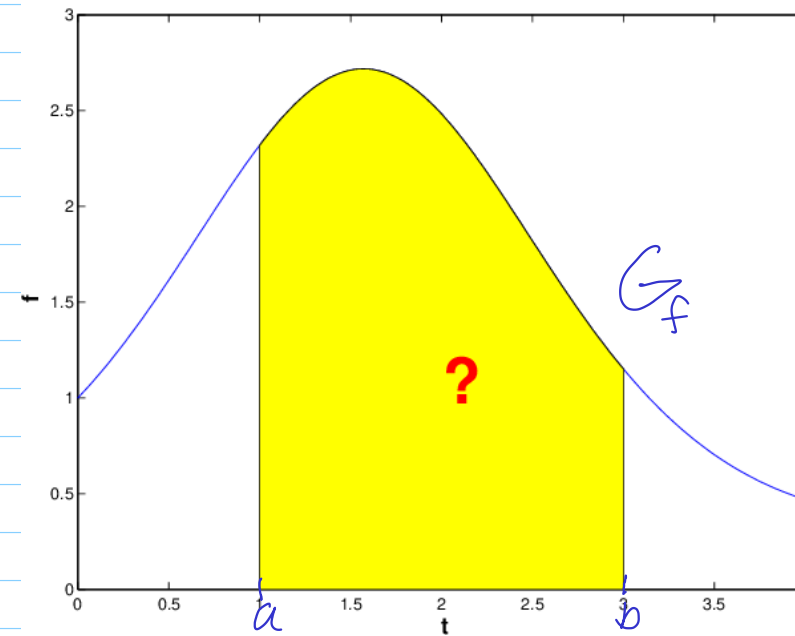


Dependencies. Section 5.2.1, Section 7.2

Duration: 5 minutes

VII Numerical Quadrature

7.4 Gauss Quadrature



Approximate
 $\int_a^b f(t) dt$

Definition 7.2.0.1. Quadrature formula/quadrature rule

An n -point quadrature formula/quadrature rule (QR) on $[a, b]$ provides an approximation of the value of an integral through a weighted sum of point values of the integrand:

$$\int_a^b f(t) dt \approx Q_n(f) := \sum_{j=1}^n w_j^n f(c_j^n). \quad (7.2.0.2)$$

7.4.1 Order of a Quadrature Rule

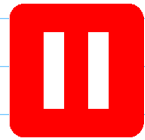
Gauge of quality of Q.R.:

Definition 7.4.1.1. Order of a quadrature rule $\leftrightarrow$ polynomial exactness

The **order** of quadrature rule $Q_n : C^0([a, b]) \rightarrow \mathbb{R}$ is defined as

$$\text{order}(Q_n) := \max\{m \in \mathbb{N}_0 : Q_n(p) = \int_a^b p(t) dt \quad \forall p \in \mathcal{P}_m\} + 1, \quad (7.4.1.2)$$

that is, as the **maximal** degree +1 of polynomials for which the quadrature rule is guaranteed to be exact.



Order of polynomial
n-pt. Q.R. = n

- Order is stable under affine trf. of Q.R.

Corollary 7.4.1.4. Invariance of order under affine transformation

An affine transformation of a quadrature rule according to Rem. 7.2.0.4 does not change its order.

Reason: $\phi : [-1, 1] \rightarrow [a, b]$ affine:

$$p \in \mathcal{P}_{n-1} \Rightarrow t \mapsto p(\phi(t)) \in \mathcal{P}_{n-1}$$

For **any** node set $\{c_1, \dots, c_n\}$
 $\exists$ weights $\{w_j\}_{j=1}^n$ that yield order- n Q.R.

$$\forall p \in \mathcal{P}_{n-1} : p = \sum_{j=1}^n p(t_j) \underset{\substack{\uparrow \\ \text{Lagrange pol.}}}{L_j}$$

$$\int_a^b p(t) dt = \sum_{j=1}^n p(t_j) \underbrace{\int_a^b L_j(t) dt}_{= \text{weight } w_j}$$

Theorem 7.4.1.6. Sufficient order conditions for quadrature rules

An n -point quadrature rule on $[a, b]$ ($\rightarrow$ Def. 7.2.0.1)

$$Q_n(f) := \sum_{j=1}^n w_j f(c_j), \quad f \in C^0([a, b]),$$

with nodes $t_j \in [a, b]$ and weights $w_j \in \mathbb{R}$, $j = 1, \dots, n$, has **order** $\geq n$, if and only if

$$w_j = \int_a^b L_{j-1}(t) dt, \quad j = 1, \dots, n,$$

where L_k , $k = 0, \dots, n-1$, is the k -th **Lagrange polynomial** (5.2.2.4) associated with the ordered node set $\{c_1, c_2, \dots, c_n\}$.

③

Rem 7.4.1.7 : (Linear system for quadrature weights)

$\{p_1, \dots, p_n\} \hat{=}$ basis of $\mathcal{P}_{n-1}$

Aim: $Q_n(p) = \int_a^b p(t) dt \quad \forall p \in \mathcal{P}_{n-1}$

$\Leftrightarrow$

$$Q_n(p_j) = \int_a^b p_j(t) dt \quad \forall j = 1, \dots, n \quad \Leftrightarrow \quad Q_n \text{ has order } \geq n. \quad (7.4.1.8)$$

$$\Leftrightarrow Q_n(f) = \sum_{j=1}^n w_j f(t_j)$$

nyn LSE:

$$\begin{bmatrix} p_1(c_1) & \dots & p_1(c_n) \\ \vdots & & \vdots \\ p_n(c_1) & \dots & p_n(c_n) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \int_a^b p_1(t) dt \\ \vdots \\ \int_a^b p_n(t) dt \end{bmatrix}. \quad (7.4.1.9)$$

↑
weights

