

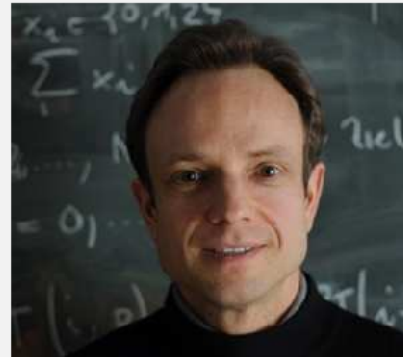
Course Video

Section 7.4.3: (Gauss-Legendre) Quadrature Error Estimates

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependencies. Section 6.2.2

Duration: 32 minutes

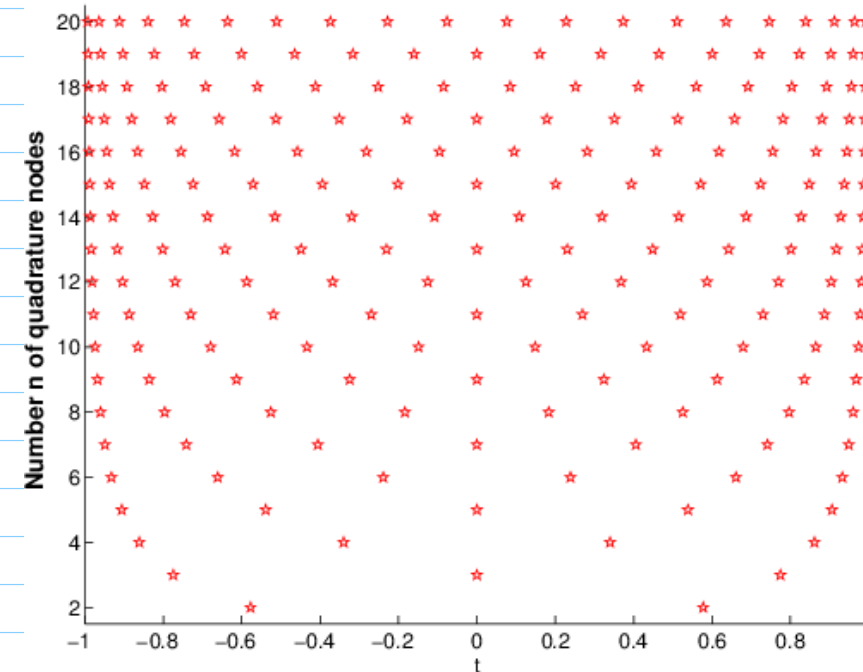
VII Numerical Quadrature

7.4 Gauss Quadrature

Definition 7.2.0.1. Quadrature formula/quadrature rule

An n -point quadrature formula/quadrature rule (QR) on $[a, b]$ provides an approximation of the value of an integral through a *weighted sum* of point values of the integrand:

$$\int_a^b f(t) dt \approx Q_n(f) := \sum_{j=1}^n w_j^n f(c_j^n). \quad (7.2.0.2)$$

Zeros of Legendre polynomials in $[-1, 1]$ 

Gauss nodes

unique n -pt Q.R.
achieving order $2n$

(2)

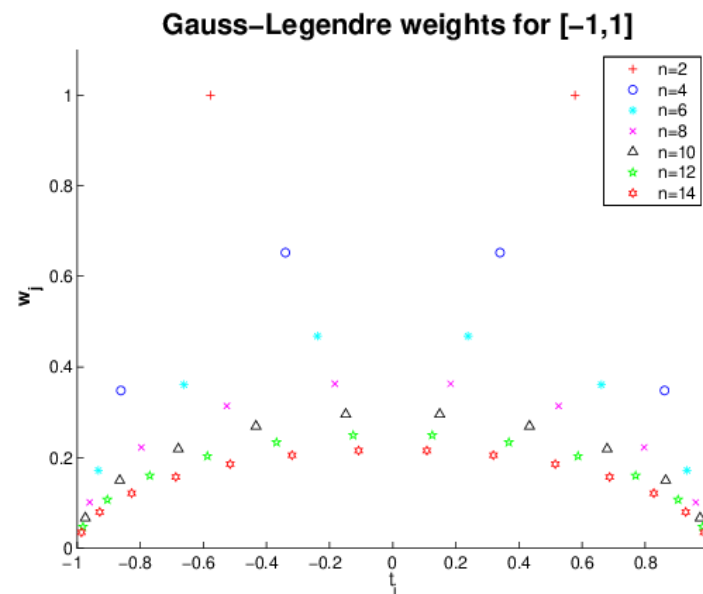
7.4.3 Quadrature Error Estimates

Obviously

Lemma 7.4.3.1. Positivity of Gauss-Legendre quadrature weights

The weights of the Gauss-Legendre quadrature formulas are positive.

▷



[also CC-rules]

Proof: [Gauss nodes $c_j^{(n)}$]

$$q_k(t) := \prod_{\substack{j=1 \\ j \neq k}}^n (t - c_j^{(n)})^2 \in \mathcal{P}_{2n-2} \quad k=1, \dots, n$$

$q_k(c_j^{(n)}) = 0 \quad j \neq k$

$$0 < \int_{-1}^1 q_k(t) dt = \sum_{j=1}^n w_j \underbrace{q_k(c_j^{(n)})}_{>0}$$

□

§ 7.4.3.2 (Quadrature error and best-approximation error)

$Q_n \hat{=}$ G.R. of order $q \geq 1$ on $[a, b]$

Quadrature error, $f \in C^0([a, b])$

$$E_n(f) = E_n(f - p) \quad \forall p \in \mathcal{P}_{q-1}$$

$$= \left| \int_a^b (f-p)(t) dt - \sum_{j=1}^n w_j (f-p)(c_j) \right|$$

$$\leq (b-a) \|f-p\|_\infty + \sum_{j=1}^n |w_j| \|f-p\|_\infty$$

$$\leq (b-a + \sum_{j=1}^n |w_j|) \|f-p\|_\infty$$

If $w_j \geq 0$ $\xRightarrow{\text{order } q \geq 1} \sum_{j=1}^n |w_j| = b-a$

$$\triangleright E_n(f) \leq 2(b-a) \|f-p\|_\infty \quad \forall p \in \mathcal{P}_{q-1}$$

③

Theorem 7.4.3.3. Quadrature error estimate for quadrature rules with positive weights

For every n -point quadrature rule Q_n as in (7.2.0.2) of order $q \in \mathbb{N}$ with weights $w_j \geq 0, j = 1, \dots, n$ the quadrature error satisfies

$$E_n(f) := \left| \int_a^b f(t) dt - Q_n(f) \right| \leq 2|b-a| \underbrace{\inf_{p \in \mathcal{P}_{q-1}} \|f - p\|_{L^\infty([a,b])}}_{\text{best approximation error}} \quad \forall f \in C^0([a,b]) \quad (7.4.3.4)$$

▷ Apply estimates of Ch. 6

Theorem 6.2.1.11. L^∞ polynomial best approximation estimate

If $f \in C^r([-1,1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then, for any polynomial degree $n \geq r$,

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} \leq (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \|f^{(r)}\|_{L^\infty([-1,1])},$$

where

$$\|f^{(r)}\|_{L^\infty([-1,1])} := \max_{x \in [-1,1]} |f^{(r)}(x)|.$$

▷ Asymptotics of $E_n(f)$ for $n \rightarrow \infty$
 [for Q, R. w/ $w_j > 0$]

• Case of limited smoothness of f :

$$f \in C^r([a,b]), \quad f \notin C^{r+1}([a,b]), \quad r < q$$

$$\Rightarrow E_n(f) = O(n^{-r}) \text{ for } n \rightarrow \infty$$

$\hat{=}$ algebraic cvg with rate r

• Analytic C^∞ -integrand

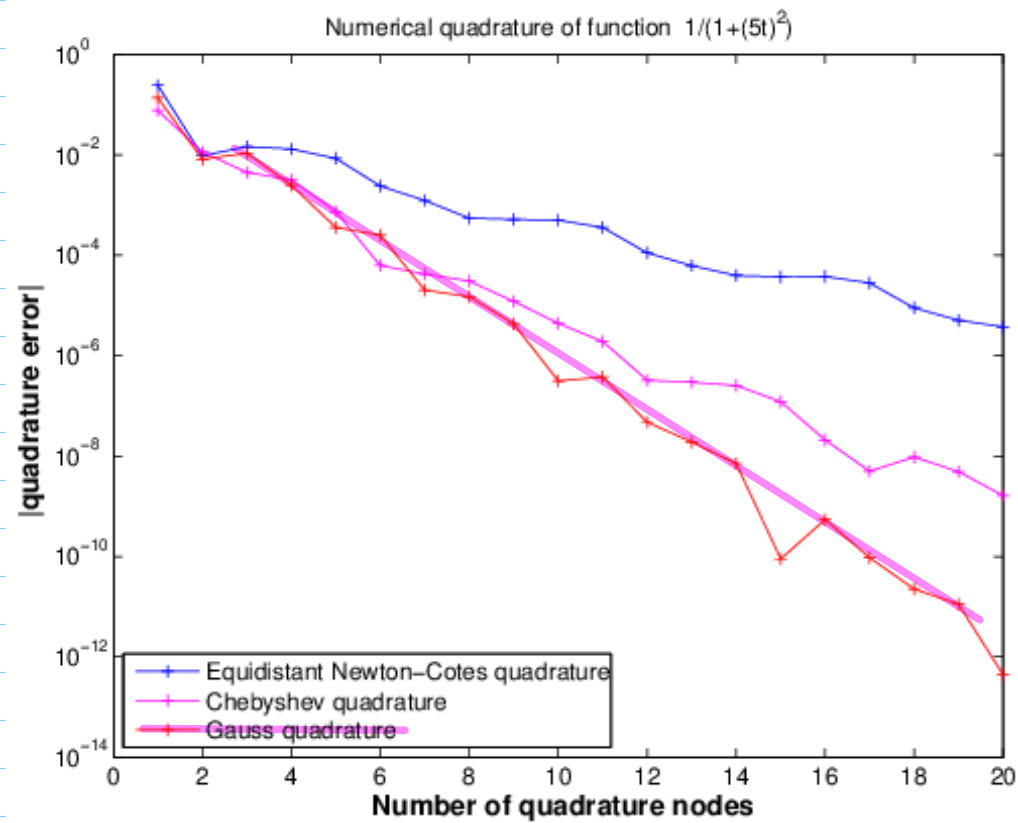
$$\exists q \in [0,1] : E_n(f) = O(q^n) \text{ for } n \rightarrow \infty$$

$\hat{=}$ exponential cvg.

Ex. 7.4.3.9: (Convergence of global quadrature rules)

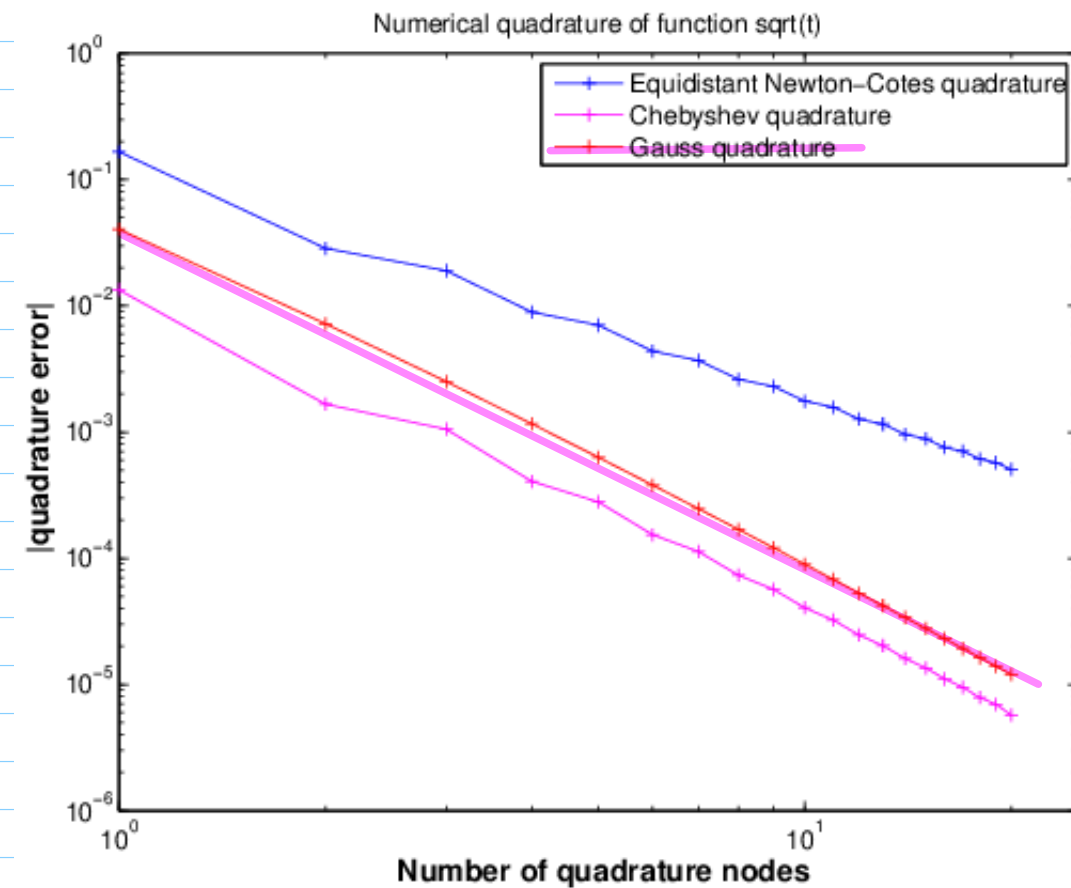
$$I = [0, 1]$$

1. $f_1(t) = \frac{1}{1+(5t)^2}$, an analytic function, see Rem. 6.2.2.59,
2. $f_2(t) = \sqrt{t}$, merely continuous, derivatives singular in $t = 0$.



lin-log plot
 $\Downarrow$
 empiric exp-cvg.

quadrature error, $f_1(t) := \frac{1}{1+(5t)^2}$ on $[0, 1]$



quadrature error, $f_2(t) := \sqrt{t}$ on $[0, 1]$

log-log plot
 $\Downarrow$
 emp. alg.
 cvg.

5 Rem. 7.4.3.10 (Removing a singularity by
 transformation)

$$I := \int_0^b \underbrace{\sqrt{t}} f(t) dt \quad \text{with } f \in C^\infty([0, b])$$

↳ non-smooth integrand

[slow alg. conv. of GL-Q.R.]

Idea: substitution $s = \sqrt{t}$ ($ds = \frac{1}{2\sqrt{t}} dt$
 $dt = 2s ds$)

$$I = \int_0^b \sqrt{t} f(t) dt = 2 \int_0^{\sqrt{b}} \underbrace{s f(s^2)}_{\in C^\infty([0, \sqrt{b}])} ds$$

[analytic]

exponential conv. of G.-L. Q.R.

Rem 7.4.3.12 (The message of asymptotic
 estimates)

What does $E_n(f) = O(n^{-r})$ tell us?

[Implicit sharpness: $E_n(f) \approx \underset{\substack{\uparrow \\ \text{unknown, elusive}}}{C} n^{-r} \quad (*)]$

▷ no information about $E_n(f)$ for concrete n

(*) tells us how much additional work $\sim n$
 is needed to reduce $E_n(f)$ by a factor $S > 1$

goal: $\frac{E_n(f)}{E_m(f)} \stackrel{!}{=} S$

$$(*) \Rightarrow \frac{C n^{-r}}{C m^{-r}} = S \Rightarrow m = n \cdot S^{1/r}$$

In the case of algebraic convergence with rate $r \in \mathbb{R}$ a reduction of the quadrature error by a factor of ρ is bought by an increase of the number of quadrature points by a factor of $\rho^{1/r}$.

⑥

Exp. conv. : $E_n(f) = O(q^n)$
 $[E_n(f) \approx C q^n \quad (+)]$

goal : $\frac{E_n(f)}{E_m(f)} \stackrel{!}{=} \rho$

$(+) \Rightarrow \frac{\cancel{C} q^n}{\cancel{C} q^m} \stackrel{!}{=} \rho \Rightarrow m - n \geq -\frac{\log \rho}{\log q}$

In the case of exponential convergence (7.4.3.15) a fixed increase of the number of quadrature points by $-\log \rho : \log q$ results in a reduction of the quadrature error by a factor of $\rho > 1$.

