

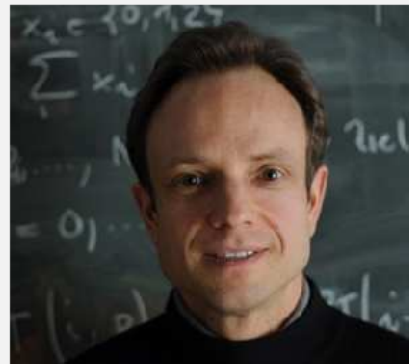
# Course Video

## Section 7.5: Composite Quadrature

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(C) Seminar für Angewandte Mathematik, ETH Zürich



**Dependencies.** Section 6.6.1

Duration:     minutes



Note: Possible minor *mismatch of video and tablet notes!*

[Corrections and updates can be incorporated into tablet notes only]

# VII Numerical Quadrature

## 7.5 Composite Quadrature

interpolation  
schemes



approximation  
schemes



quadrature  
schemes

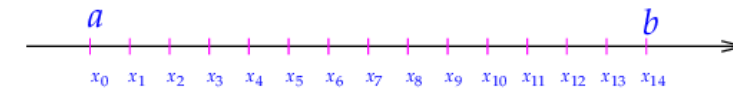
Q.R. induced by *piecewise*  
polynomial interpolation

A different view:

- Start from mesh

Terminology:

- ♦  $x_j \triangleq$  **nodes** of the mesh  $\mathcal{M}$ ,  $j = 1, \dots, m$
- ♦  $[x_{j-1}, x_j] \triangleq$  **intervals/cells** of the mesh,
- ♦  $h_{\mathcal{M}} := \max_j |x_j - x_{j-1}| \triangleq$  **mesh width**,
- ♦ If  $x_j = a + jh \triangleq$  **equidistant** (uniform) mesh with meshwidth  $h > 0$



$$\int_a^b f(t) dt = \sum_{j=1}^m \int_{x_{j-1}}^{x_j} f(t) dt. \quad (7.5.0.1)$$

↳ apply an  $n_j$ -pt local Q.R.

(2)

## General construction of composite quadrature rules



Idea: Partition integration domain  $[a, b]$  by a **mesh**/grid ( $\rightarrow$  Section 6.6)  
 $\mathcal{M} := \{a = x_0 < x_1 < \dots < x_m = b\}$

Apply quadrature formulas from Section 7.3, Section 7.4 **locally**  
 on mesh intervals  $I_j := [x_{j-1}, x_j]$ ,  $j = 1, \dots, m$ , and sum up.

**composite quadrature rule**

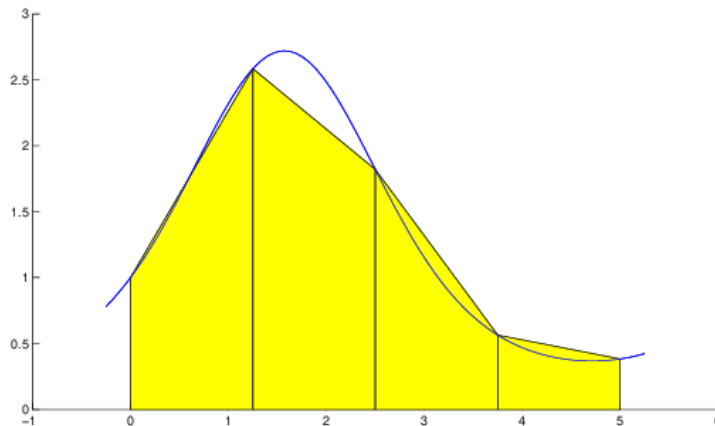
$$\text{cost} = \# \{f\text{-eval}\} \sim \sum_{j=0}^m n_j$$

Ex. 7.5.0.3 (Simple composite polynomial quadrature rules)

$$n_j = 2$$

Composite trapezoidal rule, cf. (7.3.0.5)

$$\int_a^b f(t) dt = \frac{1}{2}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \frac{1}{2}(x_{j+1} - x_{j-1})f(x_j) + \frac{1}{2}(x_m - x_{m-1})f(b). \quad (7.5.0.4)$$

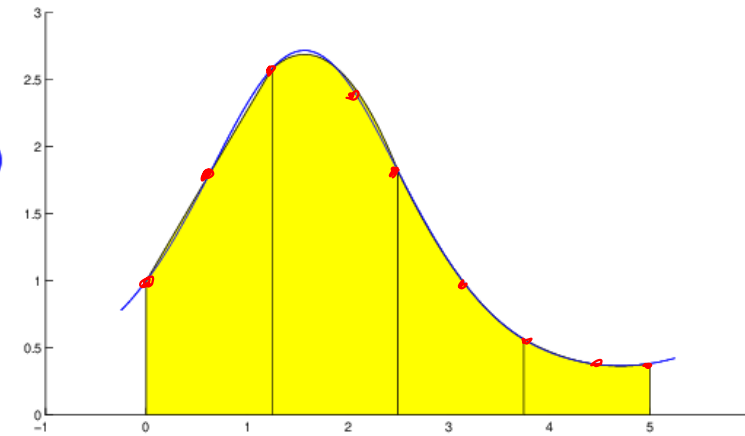


$\triangleright$  Local order = 2

$$n_j = 3$$

Composite Simpson rule, cf. (7.3.0.6)

$$\int_a^b f(t) dt = \frac{1}{6}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \frac{1}{6}(x_{j+1} - x_{j-1})f(x_j) + \sum_{j=1}^m \frac{2}{3}(x_j - x_{j-1})f\left(\frac{1}{2}(x_j + x_{j-1})\right) + \frac{1}{6}(x_m - x_{m-1})f(b). \quad (7.5.0.5)$$



$\triangleright$  Local order = 4

§ 7.5.0.9 (Quadrature error estimate for composite polynomial quadrature rules)

In  $[x_{j-1}, x_j]$ : Use **order  $q$**   $n_j$ -pts local Q.R. with weights  $> 0$ !

$\triangleright$  Q, E. on  $[x_{j-1}, x_j]$

$$\hookrightarrow \leq 2 |x_j - x_{j-1}| \inf_{p \in \mathcal{P}_{q-1}} \|f - p\|_{\infty, [x_{j-1}, x_j]}$$

③

 $f \in C^{n+1}([a, b])$ :

Thm. 6.2.2.15  $\Rightarrow$   $\|f - L_T f\|_{L^\infty(I)} \leq \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \underbrace{\max_{t \in I} |(t - t_0) \cdots (t - t_n)|}_{\leq |I|^{n+1}} \quad (6.2.2.22)$

 $f \in C^r([a, b])$ 

$$\left| \int_{x_{j-1}}^{x_j} f(t) dt - Q_{n_j}^j(f) \right| \leq C |x_j - x_{j-1}|^{\min\{r, q_j\}+1} \|f^{(\min\{r, q_j\})}\|_{L^\infty([x_{j-1}, x_j])} \quad (7.3.0.10)$$

Adding up:

$$\left| \int_{x_0}^{x_m} f(t) dt - Q(f) \right| \leq C \sum_{j=1}^m h_j^{\min\{r, q_j\}+1} \|f^{(\min\{r, q_j\})}\|_{L^\infty([x_{j-1}, x_j])},$$

$$\left| \int_{x_0}^{x_m} f(t) dt - Q(f) \right| \leq C h_{\mathcal{M}}^{\min\{q, r\}} |b - a| \|f^{(\min\{q, r\})}\|_{L^\infty([a, b])} \quad (7.5.0.10)$$

"h-convergence"

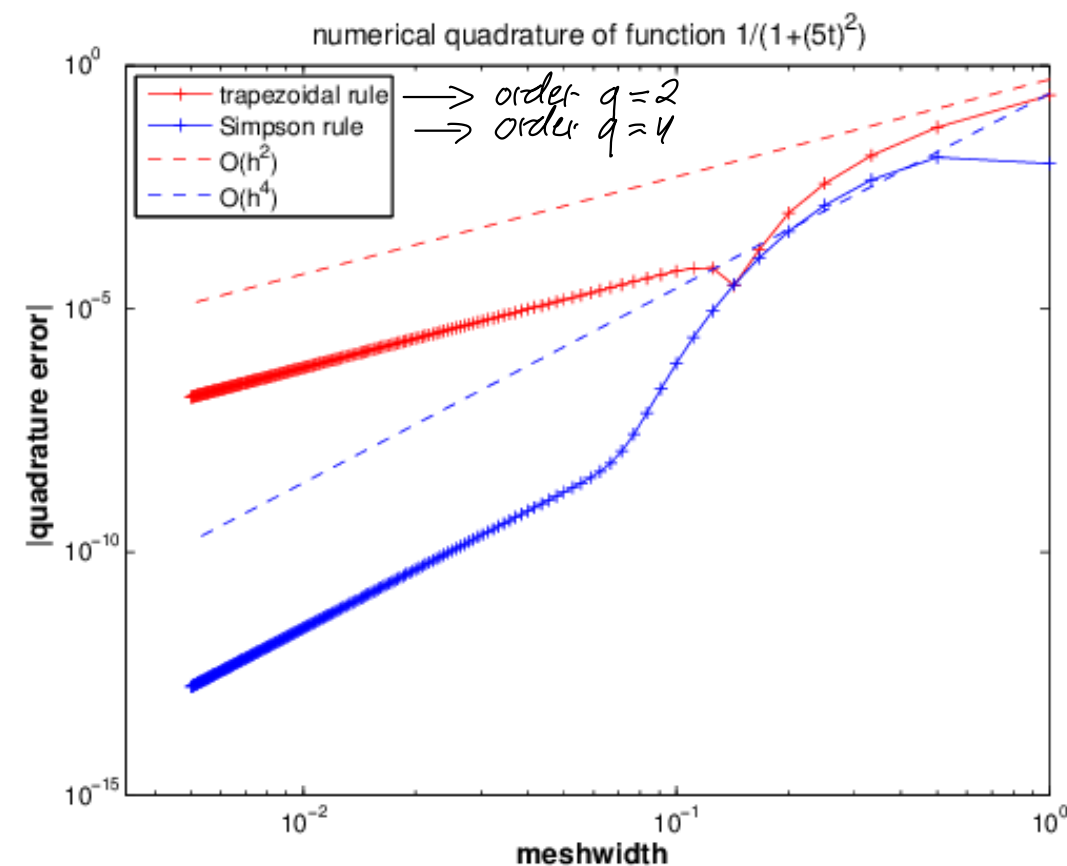
smoothness limits conv.

Algebraic conv. for  $h_m \rightarrow 0$  with rate  $\min\{q, r\}$ 

Exp.: (Quadrature error for composite quadrature rules)

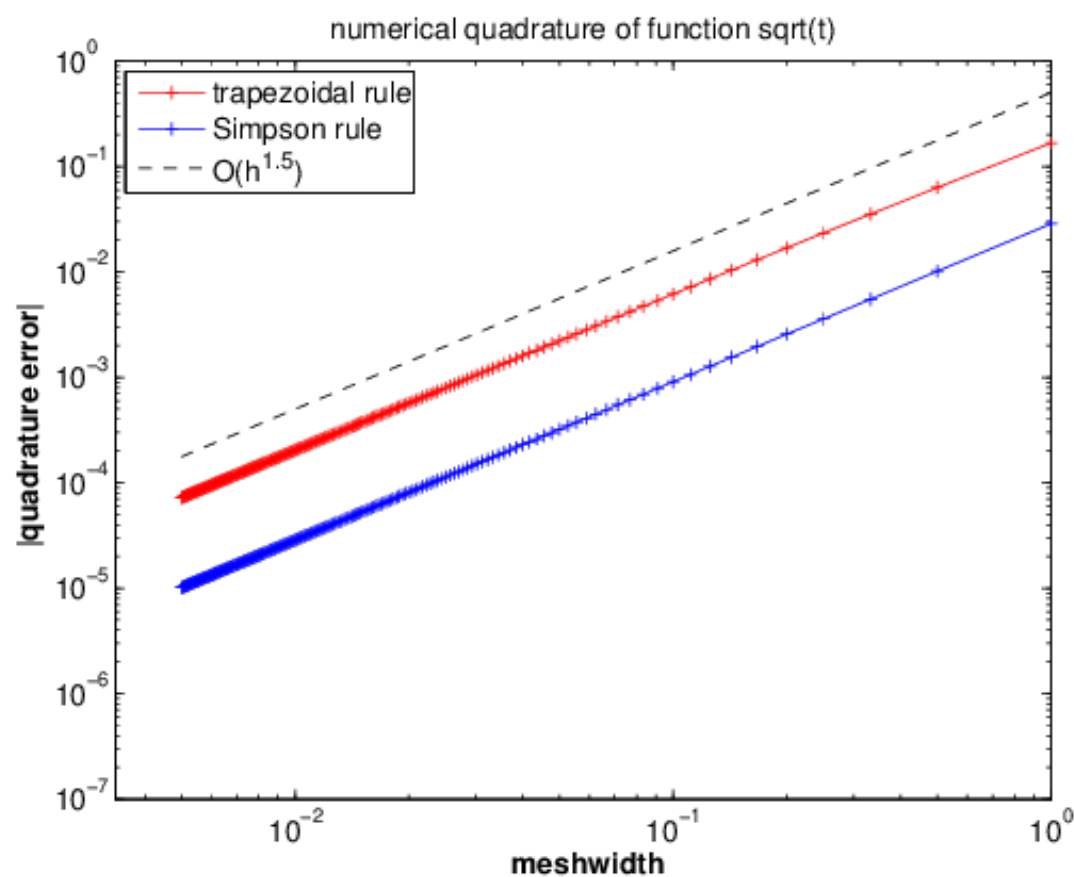
 $\triangleright$  On  $[0, 1]$ , equidistant mesh

$$x_j = j/m, \quad j = 0, \dots, m$$

 $\Delta$  log-log plot

alg. conv.

quadrature error,  $f_1(t) := \frac{1}{1+(5t)^2}$  on  $[0, 1]$  $\hookrightarrow C^\infty$ -integrand



Δ log-log plot  
↓  
alg. conv.

quadrature error,  $f_2(t) := \sqrt{t}$  on  $[0, 1]$

↳  $\notin C^1([0, 1])$ , singular

Alg. conv. in  $h_m \rightarrow 0$  with rate 1.5

Rem. 7.5.0.13 (Composite quadrature rule vs. global quadrature rules)

↪ G.-L. Q.R.  
on equidistant meshes

Comp. Q.R.

G.-L. Q.R.

$f \in C^r([a, b])$  :  $O(n^{-\min\{r, 3\}})$        $O(n^{-r})$   
[ G.-L. Q.R. is at least as good, always achieves optimal rate ]

$f \in C^\infty([a, b])$   
(analytic)       $O(n^{-q})$       exp. conv.

Use Gauss-Legendre quadrature instead of fixed order composite quadrature on equidistant meshes. \*

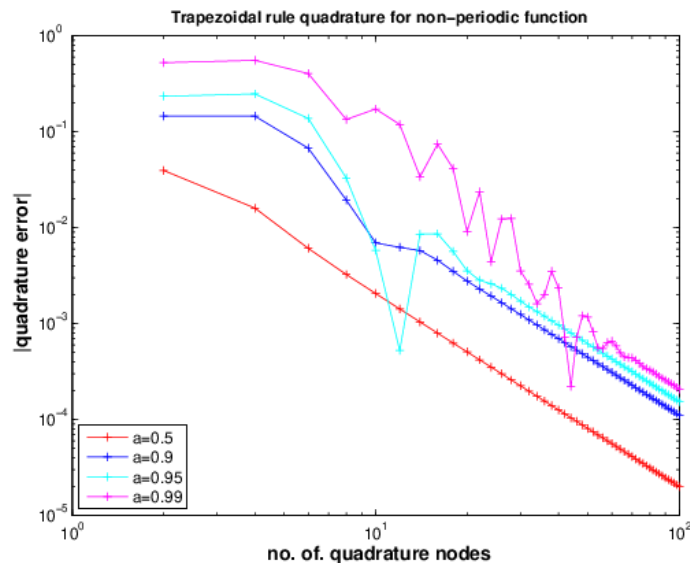
\* Comp. Q.R. may be better on locally refined meshes

# Exp. 7.5.0.16 (Empiric convergence of equidistant trapezoidal rule)

$$\int_a^b f(t) dt \approx T_m(f) := h \left( \frac{1}{2}f(a) + \sum_{k=1}^{m-1} f(kh) + \frac{1}{2}f(b) \right), \quad h := \frac{b-a}{m}. \quad (7.5.0.17)$$

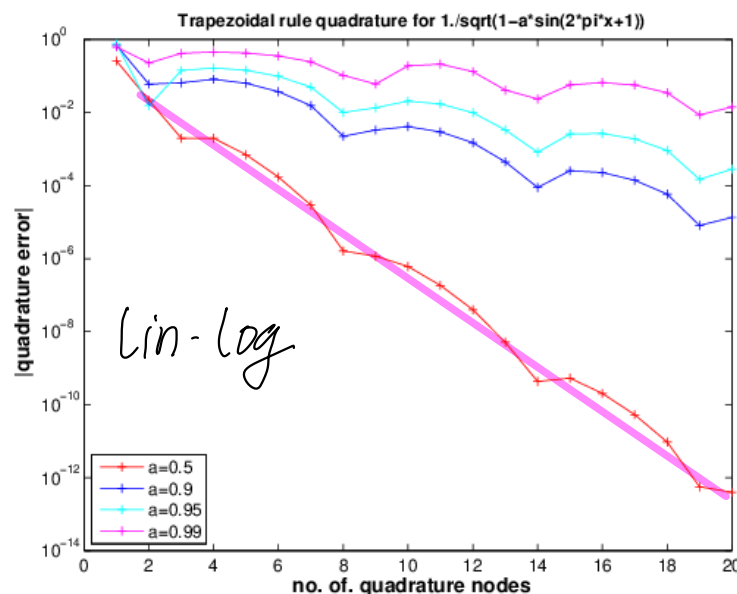
$$f(t) = \frac{1}{\sqrt{1-a \sin(2\pi t - 1)}}, \quad 0 < a < 1. \in C^\infty$$

1-periodic



quadrature error for  $T_n(f)$  on  $[0, \frac{1}{2}]$

merely algebraic convergence



quadrature error for  $T_n(f)$  on  $[0, 1]$

exponential convergence !!

# § 7.5.0.18 (The magic of the equidistant trapezoidal rule)

$f = 1$ -periodic

$$T_n(f) = \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right) \quad : \text{trapezoidal rule}$$

For  $f_k(t) = e^{2\pi i k t}$ ,  $k \in \mathbb{Z}$

$$\int_0^1 e^{2\pi i k t} dt = \delta_{0,k} \quad T_n(f_k) = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k \frac{j}{n}}$$

$$= \begin{cases} 1, & k \in n\mathbb{Z} \\ 0, & \text{else} \end{cases}$$

$$\triangleright T_n(f_k) = \int_0^1 f_k(t) dt, \quad -n < k < n$$

## Lemma 7.5.0.22. Exact quadrature by equidistant trapezoidal rule

The  $n$ -point equidistant trapezoidal quadrature rule on  $[0, 1]$

$$\int_0^1 f(t) dt \approx T_n(f) := h \left( \frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(a + kh) + \frac{1}{2}f(1) \right), \quad h := \frac{1}{n}. \quad (7.5.0.17)$$

is exact for trigonometric polynomials ( $\rightarrow$  Section 6.5.1) of degree  $\leq 2n - 2$ .

⑥

$$\mathcal{P}_{2n}^T = \text{Span} \{ t \mapsto e^{2\pi i k t}, -n \leq k \leq n \}$$

$$\triangleright \left| \int_0^1 f(t) dt - T_n(f) \right| \leq 2 \inf_{p \in \mathcal{P}_{2n-2}^T} \|f - p\|_{\infty, [0,1]}$$

best approximation  
error estimates by  
trig. pol.

#### Numerical Quadrature of *periodic integrands*

Use the equidistant trapezoidal rule for the numerical quadrature of a periodic integrand over its interval of periodicity.

$\triangleright$  Please try to answer *Review questions 7.5.0.26* without aids

⑦