

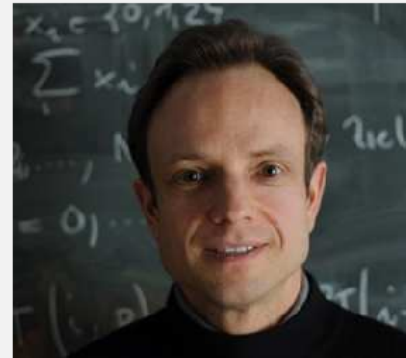
Course Video

Section 8.2.2: Iterative Methods: Speed of Convergence

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Computing with logarithms
- Norms

Dependencies. Section 8.2.1

Duration: 26 minutes

VIII Iterative Methods for Non-Linear Systems of Equations

8.2. Iterative Methods

8.2.2. Speed of Convergence

How fast $\|x^{(k)} - x^*\| \rightarrow 0$?

Definition 8.2.2.1. Linear convergence

A sequence $x^{(k)}$, $k = 0, 1, 2, \dots$, in $\mathbb{R}^n$ **converges linearly** to $x^* \in \mathbb{R}^n$,

$$\exists 0 < L < 1: \|x^{(k+1)} - x^*\| \leq L \|x^{(k)} - x^*\| \quad \forall k \in \mathbb{N}_0.$$

↑
Smallest L : rate of conv.

(2)

Rem 8.2.2.6 : (Detecting linear convergence)

Assume sharpness : $\varepsilon_k := \|x^{(k+1)} - x^*\| \approx L \|x^{(k)} - x^*\|$

$$\varepsilon_{k+1} \approx L \varepsilon_k :$$

(i) Quotients : $\varepsilon_{k+1}/\varepsilon_k \approx L < 1$

$$(ii) \quad \varepsilon_k \approx L^k \varepsilon_0$$

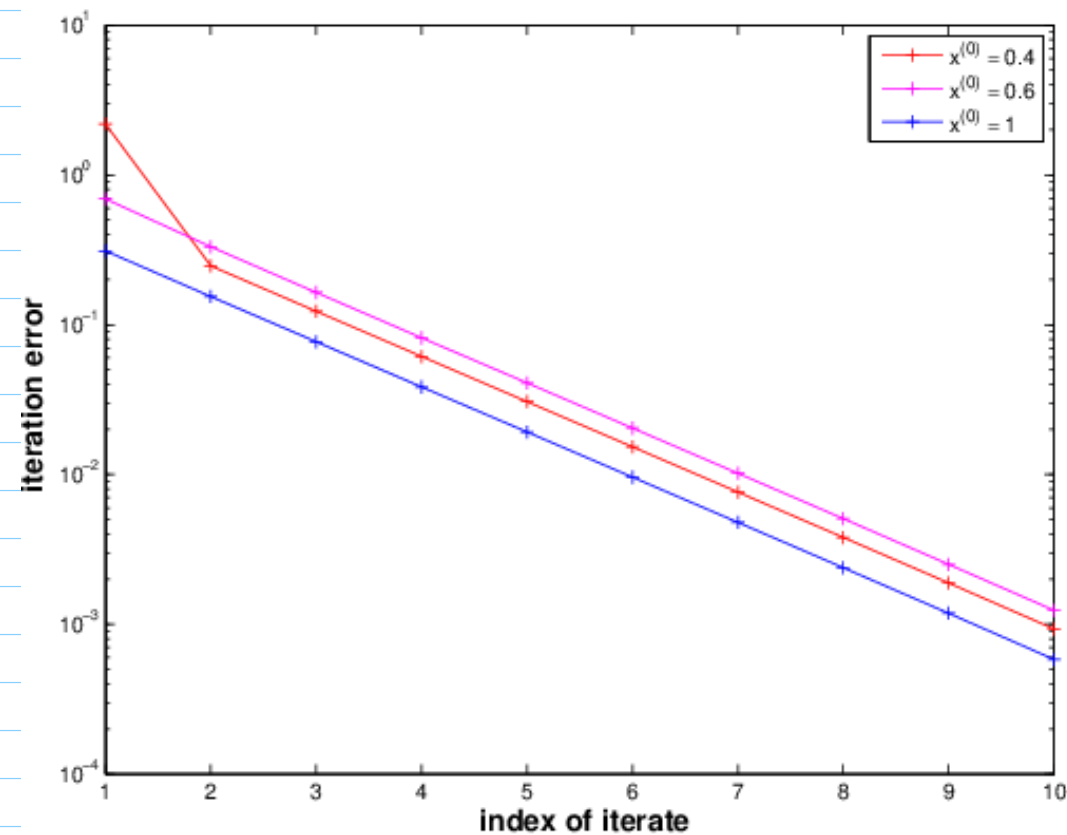
$$\log \varepsilon_k \approx k \log L + \log \varepsilon_0$$

▷ Error points aligned in lin-log plot

Ex. 8.2.2.8 (Linearly convergent iteration)

$$n=1: \quad x^{(k+1)} := x^{(k)} + \frac{\cos(x^{(k)}) + 1}{\sin(x^{(k)})}, \quad k \in \mathbb{N}_0$$

k	$x^{(0)} = 0.4$		$x^{(0)} = 0.6$		$x^{(0)} = 1$	
	$x^{(k)}$	$\frac{ x^{(k)} - x^{(15)} }{ x^{(k-1)} - x^{(15)} }$	$x^{(k)}$	$\frac{ x^{(k)} - x^{(15)} }{ x^{(k-1)} - x^{(15)} }$	$x^{(k)}$	$\frac{ x^{(k)} - x^{(15)} }{ x^{(k-1)} - x^{(15)} }$
2	3.3887	0.1128	3.4727	0.4791	2.9873	0.4959
3	3.2645	0.4974	3.3056	0.4953	3.0646	0.4989
4	3.2030	0.4992	3.2234	0.4988	3.1031	0.4996
5	3.1723	0.4996	3.1825	0.4995	3.1224	0.4997
6	3.1569	0.4995	3.1620	0.4994	3.1320	0.4995
7	3.1493	0.4990	3.1518	0.4990	3.1368	0.4990
8	3.1454	0.4980	3.1467	0.4980	3.1392	0.4980



rate ≈ 0.5

◁ lin-log plot

Definition 8.2.2.10. Order of convergence $\rightarrow$, ,

A **convergent** sequence $\mathbf{x}^{(k)}$, $k = 0, 1, 2, \dots$, in $\mathbb{R}^n$ with limit $\mathbf{x}^* \in \mathbb{R}^n$ converges with **order** p , if $p \geq 1$

$$\exists C > 0: \|\mathbf{x}^{(k+1)} - \mathbf{x}^*\| \leq C \|\mathbf{x}^{(k)} - \mathbf{x}^*\|^p \quad \forall k \in \mathbb{N}_0, \quad (8.2.2.11)$$

and, in addition, $C < 1$ in the case $p = 1$ (linear convergence $\rightarrow$ Def. 8.2.2.1).

unknown!

Rem 8.2.2.12 (Detecting order $p > 1$ of convergence)

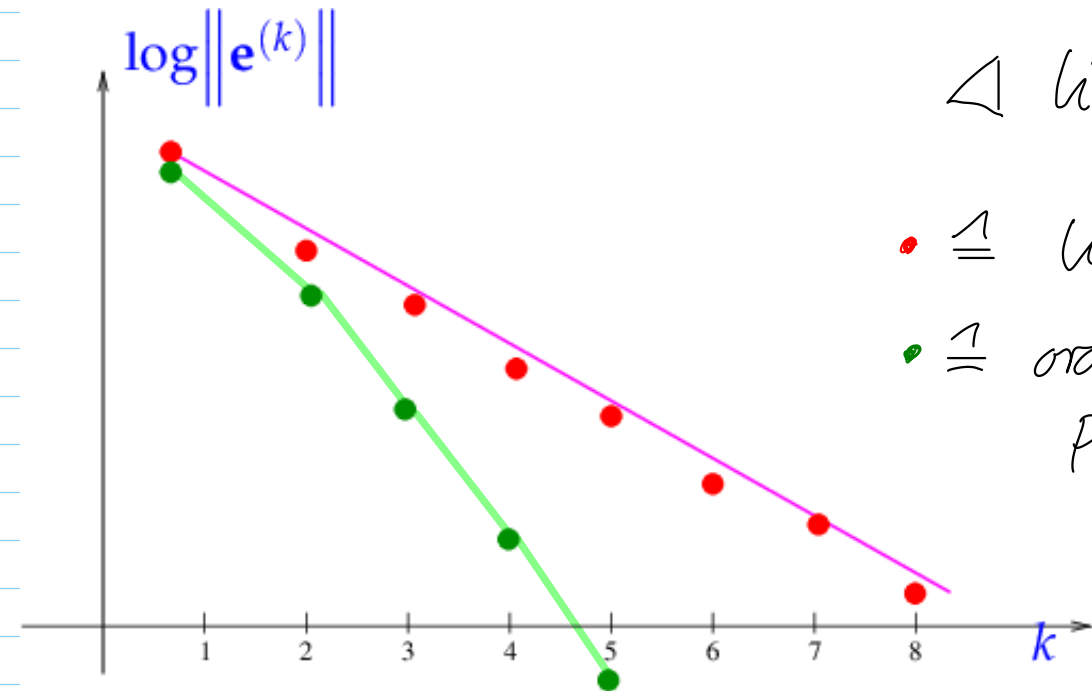
Assume sharpness : $\varepsilon_{k+1} \approx C \varepsilon_k^p$

$$\log \varepsilon_{k+1} \approx \log C + p \log \varepsilon_k$$

$$\log \varepsilon_k \approx \log C + p \log \varepsilon_{k-1}$$

$$\frac{\log \varepsilon_{k+1} - \log \varepsilon_k}{\log \varepsilon_k - \log \varepsilon_{k-1}} \approx p$$

↓
tabulate this



△ lin-log

• $\triangleq$ linear conv.

• $\triangleq$ order- p conv., $p > 1$

Ex. 8.2.2.13 (Order 2 = quadratic convergence)

$a > 0$:

$$x^{(k+1)} = \frac{1}{2} \left(x^{(k)} + \frac{a}{x^{(k)}} \right) \Rightarrow \underbrace{|x^{(k+1)} - \sqrt{a}|}_{\varepsilon_{k+1}} = \frac{1}{2x^{(k)}} \underbrace{|x^{(k)} - \sqrt{a}|^2}_{\varepsilon_k^2} \quad (8.2.2.14)$$

↓
1-point iteration

≤ $\frac{1}{2\sqrt{a}}$, since $x^{(k)} \geq \sqrt{a}$

▷ $p = 2$, $C = \frac{1}{2\sqrt{a}}$ in Def. 8.2.2.10

(4)

 $\alpha = 2$:

k	$x^{(k)}$	$e^{(k)} := x^{(k)} - \sqrt{2}$	$\log \frac{ e^{(k)} }{ e^{(k-1)} } : \log \frac{ e^{(k-1)} }{ e^{(k-2)} }$
0	2.0000000000000000	0.58578643762690485	
1	1.5000000000000000	0.08578643762690485	
2	1.4166666666666665	0.00245310429357137	1.850
3	1.4142156862745096	0.0000212390141452	1.984
4	1.4142135623746897	0.0000000000159472	2.000
5	1.4142135623730949	0.0000000000000022	0.630

Note the **doubling** of the number of correct digits in each step ! [impact of roundoff !]

$\hat{=}$ relative error 10^{-m}

$$x^{(k)} = x^* (1 + \delta_k)$$

quad. org. : $|x^{(k+1)} - x^*| \approx C |x^{(k)} - x^*|^2$

$$\Rightarrow |\delta_{k+1} x^*| \approx C |\delta_k x^*|^2$$

$$\Rightarrow \delta_{k+1} \approx \underbrace{C |x^*|}_{\text{a constant} \approx 1} \delta_k^2$$

a constant ≈ 1

▷ Review questions 8.2.2.16

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