

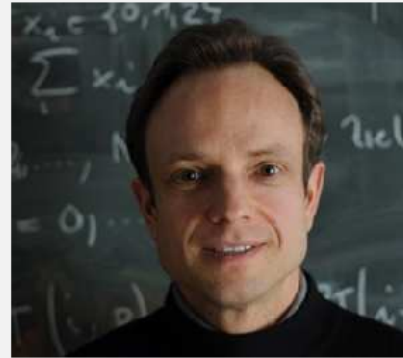
Course Video

Section 8.2.3: Iterative Methods: Termination Criteria/Stopping Rules

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Dependencies. Section 8.2.2

Duration: 25 minutes

Note: Possible minor *mismatch of video and tablet notes!*

[Corrections and updates can be incorporated into tablet notes only]

VIII Iterative Methods for Non-Linear Systems of Equations

8.2. Iterative Methods

8.2.3. Termination Criteria / Stopping Rules

Sequence $(\underline{x}^{(k)})_{k \in \mathbb{N}_0}$. Desired

$$\|\underline{x}^{(K)} - \underline{x}^*\| \leq \tau_{\text{abs}}, \quad \tau_{\text{abs}} \triangleq \text{prescribed (absolute) tolerance.}$$

or

$$\|\underline{x}^{(K)} - \underline{x}^*\| \leq \tau_{\text{rel}} \|\underline{x}^*\|, \quad \tau_{\text{rel}} \triangleq \text{prescribed (relative) tolerance.}$$

$\uparrow \triangleq \text{solution}, \quad F(\underline{x}^*) = 0$

I deal : Not accessible

$$\text{STOP at step } K = \min\{k \in \mathbb{N}_0 : \|\underline{x}^{(k)} - \underline{x}^*\| \leq \begin{cases} \tau_{\text{abs}} \\ \text{or} \\ \tau_{\text{rel}} \|\underline{x}^*\| \end{cases} \quad (8.2.3.3)$$

② § 8.2.3.4 Practical stopping rules

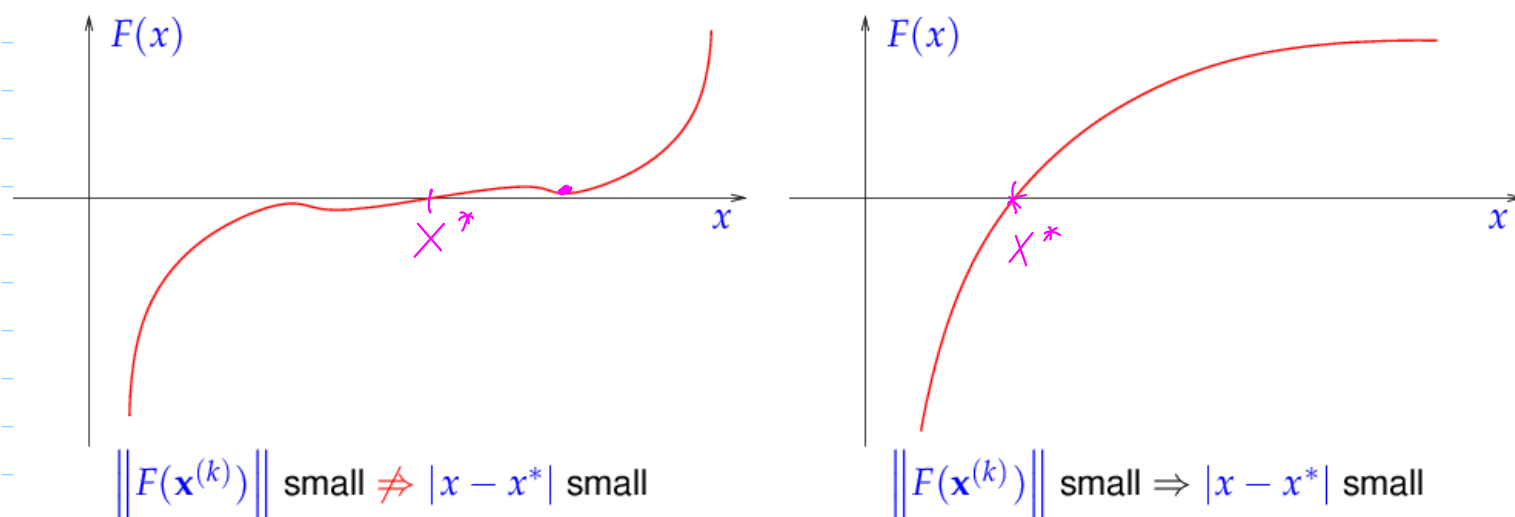
• A priori termination

→ STOP after a fixed no. of step

• Residual-based termination

STOP, when $\|F(x^{(k)})\| \leq \tau \hat{=}$ prescribed tolerance
may tell little about $\|x^{(k)} - x^*\|$

$n=1$:



• Correction-based termination

STOP convergent iteration $\{x^{(k)}\}_{k \in \mathbb{N}_0}$, when

$$\|x^{(k+1)} - x^{(k)}\| \leq \begin{cases} \tau_{\text{abs}} \\ \text{or} \\ \tau_{\text{rel}} \|x^{(k+1)}\| \end{cases}, \quad \begin{matrix} \tau_{\text{abs}} \\ \tau_{\text{rel}} \end{matrix} \text{ prescribed absolute relative tolerances } > 0.$$

↳ correction

Order p -iteration, $p > 1$: $x^{(k+1)} \approx x^*$

Rem. 8.2.3.5 (STOP, when stationary in M)

C++ code 8.2.3.6: Square root iteration →
Ex. 8.2.2.13

```

2 double sqrtit(double a)
3 {
4     double x_old = -1;
5     double x = a;
6     while (x_old != x) {
7         x_old = x;
8         x = 0.5*(x+a/x);
9     }
10    return x;
11 }

```

△ Exploit that M is finite

guaranteed rel. err.
 $\approx \text{EPS}$

③

§ 8.2.3.7 (A posteriori termination criterion for linearly convergent iterations)

$$\|x^{(k+1)} - x^*\| \leq L \|x^{(k)} - x^*\|, \quad L < 1$$

$$\|x^{(k)} - x^{(k+1)} - x^*\| \stackrel{\Delta\text{-inequ.}}{\leq} \|x^{(k+1)} - x^*\| + \|x^{(k+1)} - x^{(k)}\|$$

$$\leq L \|x^{(k)} - x^*\| + \|x^{(k+1)} - x^{(k)}\|$$

[$L < 1$]

$$(1 - L) \|x^{(k)} - x^*\| \leq \|x^{(k+1)} - x^{(k)}\|$$

$$\|x^{(k)} - x^*\| \leq \frac{L}{1 - L} \|x^{(k+1)} - x^{(k)}\|$$

$$\|x^{(k+1)} - x^*\| \leq \frac{L}{1 - L} \|x^{(k+1)} - x^{(k)}\|$$

↑
available

→ Reliable error estimate, if only we know upper bound $\bar{L}$, $L \leq \bar{L} < 1$

Ex 8.2.3.9. (A posteriori error bound for linearly convergent iteration)

$n = 1$:

$$x^{(k+1)} = x^{(k)} + \frac{\cos x^{(k)} + 1}{\sin x^{(k)}} \Rightarrow x^{(k)} \rightarrow \pi \text{ for } x^{(0)} \text{ close to } \pi.$$

Observed: $L \approx 1/2$

k	$ x^{(k)} - \pi $	$\frac{L}{1-L} x^{(k)} - x^{(k-1)} $	slack of bound
1	2.191562221997101	4.933154875586894	2.741592653589793
2	0.247139097781070	1.944423124216031	1.697284026434961
3	0.122936737876834	0.124202359904236	0.001265622027401
4	0.061390835206217	0.061545902670618	0.000155067464401
5	0.030685773472263	0.030705061733954	0.000019288261691
6	0.015341682696235	0.015344090776028	0.000002408079792
7	0.007670690889185	0.007670991807050	0.000000300917864
8	0.003835326638666	0.003835364250520	0.000000037611854
9	0.001917660968637	0.001917665670029	0.000000004701392
10	0.000958830190489	0.000958830778147	0.000000000587658
11	0.000479415058549	0.000479415131941	0.000000000073392
12	0.000239707524646	0.000239707533903	0.000000000009257
13	0.000119853761949	0.000119853762696	0.000000000000747
14	0.000059926881308	0.000059926880641	0.000000000000667
15	0.000029963440745	0.000029963440563	0.000000000000181

(4)

▷ Review questions 8.2.3.10

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⑦