

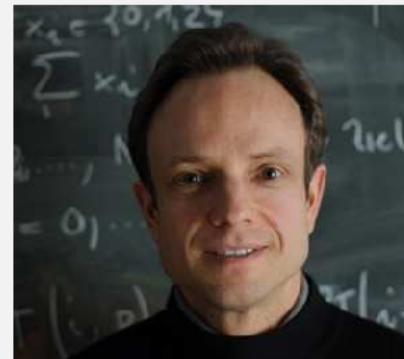
Course Video

Section 8.3: Fixed-Point Iterations

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(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Mean-value theorem
- Taylor expansion

Dependencies. Section 8.2.1, Section 8.2.2

Duration: 20 minutes

VIII Iterative Methods for Non-Linear Systems of Equations

8.3. Fixed-Point Iterations

$$NLSE : F(\underline{x}) = 0, \quad F: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Fixed-point iteration $\hat{=}$ 1-point iteration

► A fixed point iteration is defined by iteration function $\Phi: U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$:

iteration function $\Phi: U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$
initial guess $\underline{x}^{(0)} \in U$

iterates $(\underline{x}^{(k)})_{k \in \mathbb{N}_0}$: $\underline{x}^{(k+1)} := \Phi(\underline{x}^{(k)})$

→ 1-point method, cf. (8.2.1.5)

Name :

$$\underline{x}^* = \lim_{k \rightarrow \infty} \underline{x}^{(k)} \quad \xRightarrow{\Phi \text{ continuous}}$$

$$\underline{x}^* = \Phi(\underline{x}^*)$$

$\underline{x}^*$ is a fixed point of Φ

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8.3.1. Consistent Fixed-Point Iterations

Definition 8.3.1.1. Consistency of fixed point iterations, c.f. Def. 8.2.1.7

A fixed point iteration $\mathbf{x}^{(k+1)} = \Phi(\mathbf{x}^{(k)})$ is **consistent** with $F(\mathbf{x}) = 0$, if, for $\mathbf{x} \in U \cap D$,

$$F(\mathbf{x}) = 0 \quad \Leftrightarrow \quad \Phi(\mathbf{x}) = \mathbf{x}.$$

General construction of fixed point iterations that is consistent with $F(\mathbf{x}) = 0$:

- ❶ Rewrite equivalently $F(\mathbf{x}) = 0 \Leftrightarrow \Phi(\mathbf{x}) = \mathbf{x}$ and then
- ❷ use the fixed point iteration

$$\mathbf{x}^{(k+1)} := \Phi(\mathbf{x}^{(k)}). \quad (8.3.1.2)$$

→ Many possibilities !

Exp. 8.3.1.3 (Consistent fixed-point iterations)

$n = 1$:

$$F(x) = xe^x - 1, \quad x \in [0, 1].$$

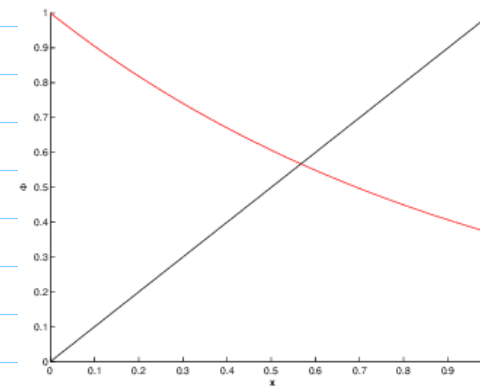
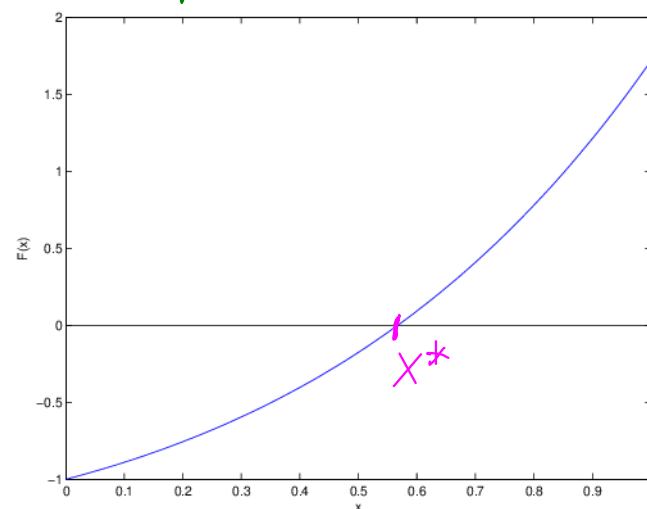
Different fixed-point forms:

$$\Phi_1(x) = e^{-x},$$

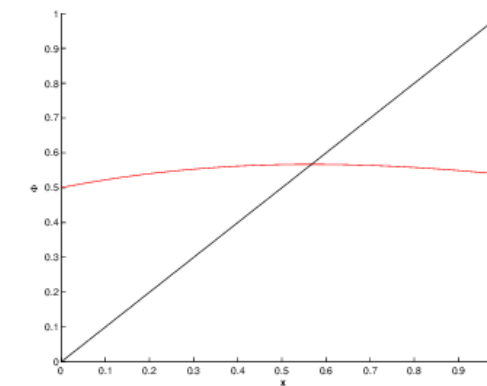
$$\Phi_2(x) = \frac{1+x}{1+e^x},$$

$$\Phi_3(x) = x + 1 - xe^x.$$

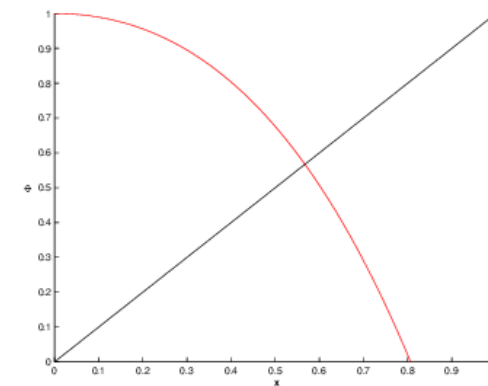
$$\phi_i(x) = x \Leftrightarrow F(x) = 0$$



function Φ_1



function Φ_2



function Φ_3

k	$ x_1^{(k+1)} - x^* $	$ x_2^{(k+1)} - x^* $	$ x_3^{(k+1)} - x^* $
0	0.067143290409784	0.067143290409784	0.067143290409784
1	0.039387369302849	0.000832287212566	0.108496074240152
2	0.021904078517179	0.000000125374922	0.219330611898582
3	0.012559804468284	0.0000000000000003	0.288178118764323
4	0.007078662470882	0.0000000000000000	0.723649245792953
5	0.004028858567431	0.0000000000000000	0.410183132337935
6	0.002280343429460	0.0000000000000000	1.186907542305364
7	0.001294757160282	0.0000000000000000	0.146569797006362
8	0.000733837662863	0.0000000000000000	0.310516641279937
9	0.000416343852458	0.0000000000000000	0.357777386500765
10	0.000236077474313	0.0000000000000000	0.974565695952037

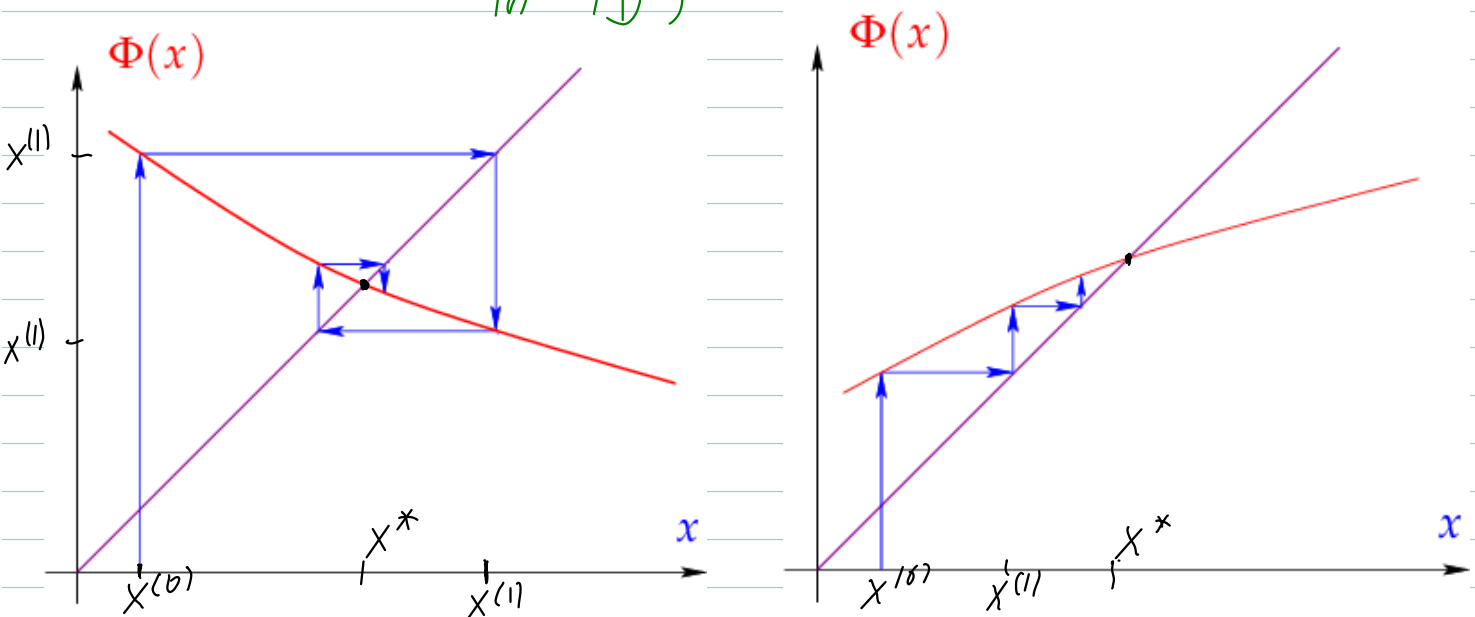
↓
linear conv.

↓
quadratic conv.

↓
no conv.

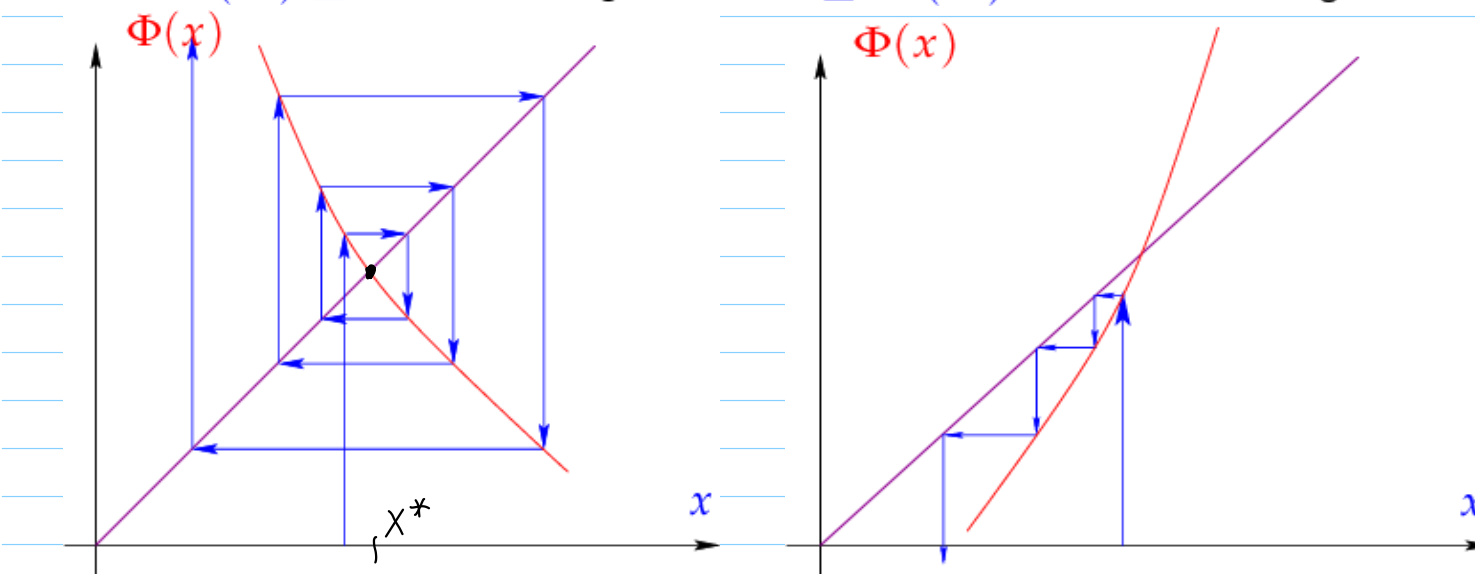
③ 8.3.2. Convergence of Fixed-Point Iterations

Rem 8.3.2.2 : (Visualization of fixed-point iterations in $\mathbb{D}$)



$-1 < \Phi'(x^*) \leq 0 \Rightarrow$ convergence

$0 \leq \Phi'(x^*) < 1 \Rightarrow$ convergence

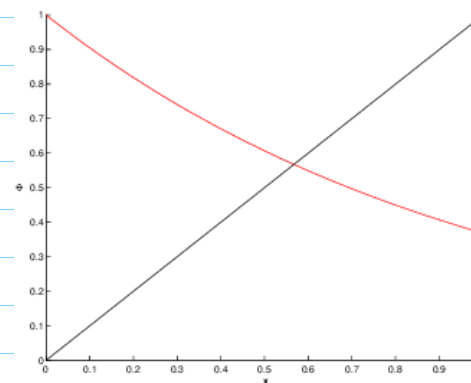


$\Phi'(x^*) < -1 \Rightarrow$ divergence

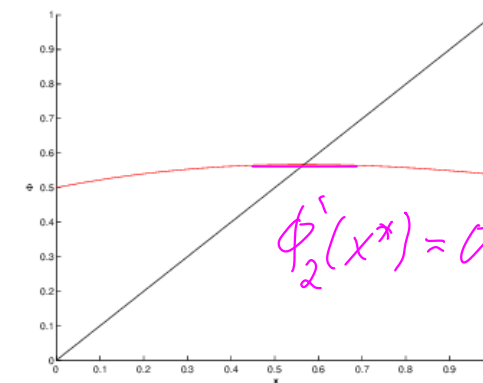
$1 < \Phi'(x^*) \Rightarrow$ divergence

$\triangleright \Phi'(x^*)$ is crucial $\leftrightarrow$ slope G_Φ

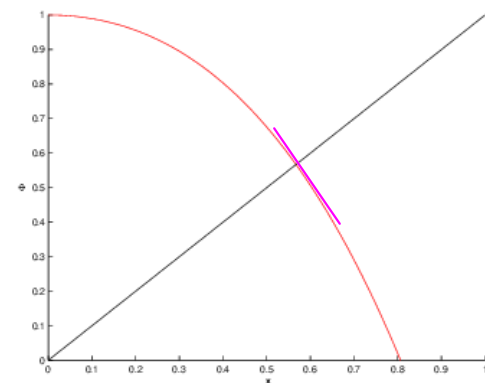
Exp 8.3.1.3 :



Φ_1 : linear convergence ?



Φ_2 : quadratic convergence ?



function Φ_3 : no convergence

Lemma 8.3.2.7. Sufficient condition for local linear convergence of fixed point iteration $\rightarrow$

If $\Phi : U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$, $\Phi(x^*) = x^*$, Φ differentiable in x^* , and $\|D\Phi(x^*)\| < 1$, then the fixed point iteration

$$x^{(k+1)} := \Phi(x^{(k)}), \quad (8.3.1.2)$$

converges locally and at least linearly that is

matrix norm, Def. 1.5.5.10 !

$$\exists 0 \leq L < 1: \|x^{(k+1)} - x^*\| \leq L \|x^{(k)} - x^*\| \quad \forall k \in \mathbb{N}_0,$$

provided that the initial guess $x^{(0)}$ belongs to some neighborhood of x^* .

$x^{(0)}$ "sufficiently close" to x^*

(4)

$$\begin{aligned}\|x^{(k+1)} - x^*\| &= \|\phi(x^{(k)}) - \phi(x^*)\| \\ &= \|\mathcal{D}\phi(x^*)(x^{(k)} - x^*) + O(\|x^{(k)} - x^*\|^2)\|\end{aligned}$$

▷ Review questions 8.3.2.21

Lemma 8.3.2.15. Higher order local convergence of fixed point iterations

If $\Phi : U \subset \mathbb{R} \mapsto \mathbb{R}$ is $m+1$ times continuously differentiable, $\Phi(x^*) = x^*$ for some x^* in the interior of U , and $\Phi^{(l)}(x^*) = 0$ for $l = 1, \dots, m$, $m \geq 1$, then the fixed point iteration (8.3.1.2) converges *locally* to x^* with *order* $\geq m+1$ ($\rightarrow$ Def. 8.2.2.10).

$$\downarrow$$

$$\Phi'(x^*) = 0 \Rightarrow \text{At least quadratic} \\ \text{cvg.}$$

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