

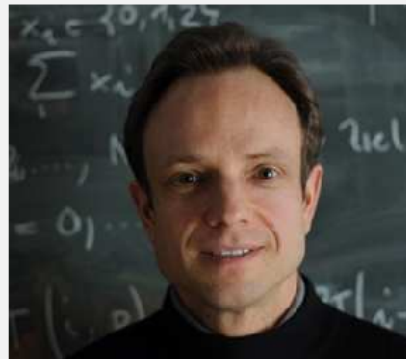
Course Video

Section 8.4.3: Asymptotic Efficiency of Iterative Methods for Zero Finding

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Dependencies. Section 8.2.1, Section 8.2.2, Section 8.4.2.3

Duration: 17 minutes

VIII Iterative Methods for Non-Linear Systems of Equations

8.4. Finding Zeros of Scalar Functions

Seek $x^* \in I : F(x^*) = 0$, $F: I \subset \mathbb{R} \rightarrow \mathbb{R}$

8.4.3. Asymptotic Efficiency of Iterative Methods for Zero Finding

$$\text{Efficiency} = \frac{\text{gain} \leftarrow \text{no. of correct digits}}{\text{work} \leftarrow \# F\text{-eval} + \# F'\text{-eval}}$$

For an iterative method ($e^{(k)} \triangleq$ iteration error)

$$\text{Gain: } |e^{(k)}| \leq \rho |e^{(0)}| \text{ for } \rho \ll 1 \quad (*)$$

$$\text{work} \leftrightarrow \underset{(*)}{\triangleq} k(\rho) \text{ minimal no. of steps it takes to achieve}$$

(2)

Notice: $|\log \rho| \leftrightarrow$ Gain in no. of significant digits of $x^{(k)}$

Measure for efficiency:

$$\text{Efficiency} := \frac{\text{no. of digits gained}}{\text{total work required}} = \frac{|\log \rho|}{k(\rho) \cdot W}$$

$W \triangleq \# F\text{-eval} + \# F'\text{-eval per step}$

Focus: order- p iteration, $p > 1$

$$\begin{aligned} |e^{(k)}| &\leq C |e^{(k-1)}|^p \leq C^{1+p} |e^{(k-2)}|^{p^2} \leq C^{1+p+p^2} |e^{(k-3)}|^{p^3} \leq \dots \\ &\leq C^{1+p^2+p^3+\dots+p^{k-1}} |e^{(0)}|^{p^k} = \underbrace{C^{\frac{p^k-1}{p-1}}}_{\text{Assume } < 1} |e^{(0)}|^{p^k-1} |e^{(0)}|, \quad k \in \mathbb{N}, \end{aligned}$$

$$|e^{(k)}| \leq \rho |e^{(0)}|, \text{ if } (C^{\frac{1}{1-p}} |e^{(0)}|)^{p^k-1} \leq \rho$$

$$k(\rho) \geq \frac{\log(1 + \frac{\log \rho}{\log L_0})}{\log p}, \quad L_0 := C^{1/p-1} |e^{(0)}| < 1 \quad (8.4.3.10)$$

§ 8.4.3.11 (Asymptotic efficiency)

$$\rho \ll 1 \Rightarrow |\log \rho| \gg 1$$

► asymptotic efficiency for $\rho \ll 1 \quad (\rightarrow |\log \rho| \rightarrow \infty)$:

$$\text{Efficiency}_{|\rho \ll 1} = \begin{cases} -\frac{\log C}{W} & , \text{ if } p = 1, \\ \frac{\log p}{W} \cdot \frac{|\log \rho|}{\log(|\log \rho|)} & , \text{ if } p > 1. \end{cases} \quad (8.4.3.12)$$

specific for the iteration: gauge for asymptotic efficiency

§ 8.4.3.14 (Comparison of Newton's method and of the secant method)

$$\begin{aligned} W_{\text{Newton}} &= 1 \text{ F-eval} + 1 \text{ F'-eval} \\ W_{\text{secant}} &= 1 \text{ F-eval} \end{aligned}$$

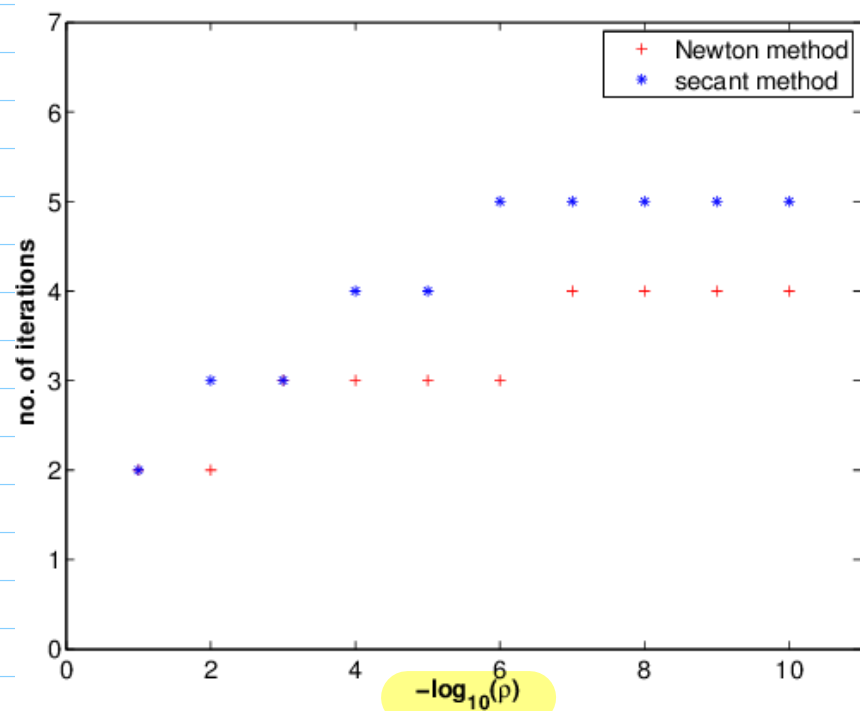
$$W_{\text{Newton}} = 2W_{\text{secant}},$$

$$p_{\text{Newton}} = 2, \quad p_{\text{secant}} = 1.62$$

$$\frac{\log p_{\text{Newton}}}{W_{\text{Newton}}} : \frac{\log p_{\text{secant}}}{W_{\text{secant}}} = 0.71.$$

▷ Secant method more efficient

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"Simulated iterations"

$$C = 7$$

$$|e^{(0)}| = 0.1$$

▷ Review questions 8.4.3.15

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