

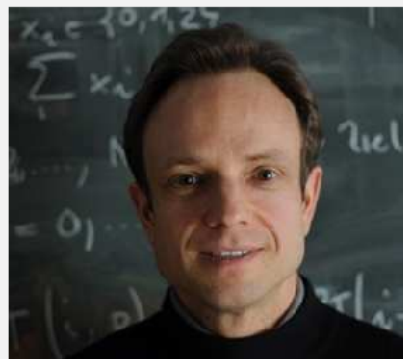
Course Video

Section 8.5.2: Convergence of Newton's Method

Prof. R. Hiptmair, SAM, ETH Zurich

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Dependencies. Section 8.5.1

Duration: 16 minutes

VIII Iterative Methods for Non-Linear Systems of Equations

8.5 Newton's Method in $\mathbb{R}^n$

NLSE: Given $C^1 F : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n \in \mathbb{N}$,
find $\underline{x}^* \in \mathcal{I} : F(\underline{x}^*) = 0$

► **Newton iteration:** (generalizes (8.4.2.1) to $n > 1$)

$$\underline{x}^{(k+1)} := \underline{x}^{(k)} - DF(\underline{x}^{(k)})^{-1}F(\underline{x}^{(k)}) \quad , \quad [\text{if } DF(\underline{x}^{(k)}) \text{ regular}] \quad (8.5.1.6)$$

Terminology: $-DF(\underline{x}^{(k)})^{-1}F(\underline{x}^{(k)})$ is called the **Newton correction**

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8.5.2. Convergence of Newton's Method

$n = 1 \Rightarrow$ local quadratic conv.

Definition 8.2.2.10. Order of convergence $\rightarrow$, ,

A **convergent** sequence $\mathbf{x}^{(k)}$, $k = 0, 1, 2, \dots$, in $\mathbb{R}^n$ with limit $\mathbf{x}^* \in \mathbb{R}^n$ converges with **order** p , $p \geq 1$, if

$$\exists C > 0: \|\mathbf{x}^{(k+1)} - \mathbf{x}^*\| \leq C \|\mathbf{x}^{(k)} - \mathbf{x}^*\|^p \quad \forall k \in \mathbb{N}_0, \quad (8.2.2.11)$$

and, in addition, $C < 1$ in the case $p = 1$ (linear convergence $\rightarrow$ Def. 8.2.2.1).

N.I. $\Leftrightarrow$ *fixed-point iteration* in $\mathbb{R}^n$

$$\underline{x}^{(k+1)} = \phi(\underline{x}^{(k)}), \quad \phi(x) = x - DF(x)^{-1} F(x)$$

$$[F(\underline{x}^*) = 0 \Leftrightarrow \phi(\underline{x}^*) = \underline{x}^*]$$

$$\|\underline{x}^{(k+1)} - \underline{x}^*\| = \|\phi(\underline{x}^{(k+1)}) - \phi(\underline{x}^*)\|$$

Taylor, $F \in C^2$ $= \|\mathcal{D}\phi(\underline{x}^*)(\underline{x}^{(k)} - \underline{x}^*) + O(\|\underline{x}^{(k)} - \underline{x}^*\|^2)\|$

By product rule:

$$\mathcal{D}\phi(\underline{x}) \underline{h} = \underline{h} - \mathcal{D}\{ \underline{x} \mapsto DF(\underline{x})^{-1} \}(\underline{x}) \underline{h} \cdot F(\underline{x}) + DF(\underline{x})^{-1} \mathcal{D}F(\underline{x}) \underline{h}$$

$$\begin{aligned} \mathcal{D}\phi(\underline{x}^*) \underline{h} &= \underline{h} - \mathcal{D}\{ \underline{x} \mapsto DF(\underline{x})^{-1} \}(\underline{x}^*) \underline{h} \cdot F(\underline{x}^*) + DF(\underline{x}^*)^{-1} \mathcal{D}F(\underline{x}^*) \underline{h} \\ &= 0 \quad \forall \underline{h} \end{aligned}$$

$$\begin{aligned} \triangleright \quad \|\underline{x}^{(k+1)} - \underline{x}^*\| &= O(\|\underline{x}^{(k)} - \underline{x}^*\|^2) \\ &\stackrel{!}{=} \text{local quadratic conv.} \end{aligned}$$

Ex. 8.5.2.3 (Convergence of Newton iteration for matrix inversion $\rightarrow$ Ex. 8.5.1.38)

$A \in \mathbb{R}^{n,n}$ regular:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (\mathbf{x}^{(k)} \mathbf{A} \mathbf{x}^{(k)} - \mathbf{x}^{(k)}) = \mathbf{x}^{(k)} (2\mathbf{I} - \mathbf{A} \mathbf{x}^{(k)}) \quad (8.5.1.42)$$

$\triangleright$ Recursion for error $\mathbf{E}^{(k)} := \mathbf{x}^{(k)} - \mathbf{A}^{-1} \in \mathbb{R}^{n,n}$

$$\mathbf{E}^{(k+1)} = \mathbf{x}^{(k+1)} - \mathbf{A}^{-1}$$

$$\stackrel{(8.5.1.42)}{=} \mathbf{x}^{(k)} (2\mathbf{I} - \mathbf{A} \mathbf{x}^{(k)}) - \mathbf{A}^{-1}$$

$$= (\mathbf{E}^{(k)} + \mathbf{A}^{-1}) (2\mathbf{I} - \mathbf{A}(\mathbf{E}^{(k)} + \mathbf{A}^{-1})) - \mathbf{A}^{-1}$$

$$= (\mathbf{E}^{(k)} + \mathbf{A}^{-1}) (\mathbf{I} - \mathbf{A} \mathbf{E}^{(k)}) - \mathbf{A}^{-1} = -\mathbf{E}^{(k)} \mathbf{A} \mathbf{E}^{(k)}.$$

$$\|AB\| \leq \|A\| \|B\|:$$

$$\blacktriangleright \quad \|\mathbf{E}^{(k+1)}\| \leq \|\mathbf{E}^{(k)}\|^2 \|\mathbf{A}\| \rightarrow \text{local quadratic conv.} \quad (8.5.2.4)$$

③

N.I. convergent:

$$\|E^{(0)}A\| = \|X^{(0)}A - I\| < 1$$

▷ Review questions 8.5, 2.8

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