

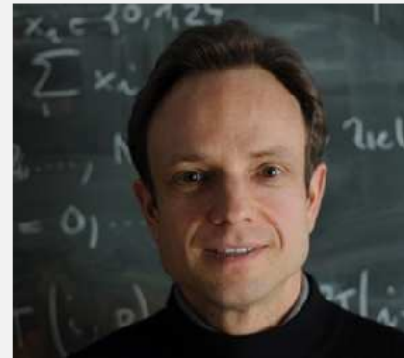
Course Video

Section 8.5.3: Termination of Newton Iteration

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Dependencies. Section 8.5.1

Duration: 12 minutes

VIII Iterative Methods for Non-Linear Systems of Equations

8.5 Newton's Method in $\mathbb{R}^n$

NLSE: Given C^1 $F : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n \in \mathbb{N}$,
find $\underline{x}^* \in \mathcal{I}$: $F(\underline{x}^*) = 0$

► **Newton iteration:** (generalizes (8.4.2.1) to $n > 1$)

$$\underline{x}^{(k+1)} := \underline{x}^{(k)} - DF(\underline{x}^{(k)})^{-1}F(\underline{x}^{(k)}) \quad , \quad [\text{if } DF(\underline{x}^{(k)}) \text{ regular}] \quad (8.5.1.6)$$

Terminology: $-DF(\underline{x}^{(k)})^{-1}F(\underline{x}^{(k)})$ is called the **Newton correction**

▷ Local **quadratic convergence** to
 $\underline{x}^* \in \mathcal{D}$, $F(\underline{x}^*) = 0$, if $DF(\underline{x}^*)$ regular

② 8.5.3. Termination of Newton Iteration

Goal: $\|x^{(N)} - x^*\|$ "small"

Quad org. $\Rightarrow$ If N.I. converges, asymptotically

$$\|x^{(k+1)} - x^*\| \ll \|x^{(k)} - x^*\|$$

$$\Rightarrow \|x^{(k)} - x^*\| \approx \|x^{(k)} - x^{(k+1)}\|$$

STOP, as soon as $\|\Delta x^{(k)}\| \leq \tau_{\text{rel}} \|x^{(k)}\|$ or $\|\Delta x^{(k)}\| \leq \tau_{\text{abs}}$,

with **Newton correction** $\Delta x^{(k)} := DF(x^{(k)})^{-1}F(x^{(k)})$.

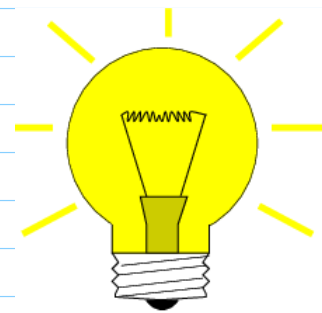
(8.5.3.2)

$\rightarrow x^{(k)}$ is already accurate enough
needlessly computed $x^{(k+1)}$: significant additional cost*

$\rightarrow$ very few steps needed

* cost = $O(n^3)$ per step

§ 8.5.3.4 (Termination of Newton iteration based on simplified Newton correction)



Idea: Replace $DF(x^{(k)})$ with $DF(x^{(k-1)})$ in any correction based termination criterion.

Rationale: LU-decomposition of $DF(x^{(k-1)})$ is already available $\rightarrow$ less effort.

Terminology:

$$\Delta \bar{x}^{(k)} := DF(x^{(k-1)})^{-1}F(x^{(k)}) \triangleq \text{simplified Newton correction}$$

► Economical correction based termination criterion for Newton's method:

STOP, as soon as $\|\Delta \bar{x}^{(k)}\| \leq \tau_{\text{rel}} \|x^{(k)}\|$ or $\|\Delta \bar{x}^{(k)}\| \leq \tau_{\text{abs}}$,

(8.5.3.5)

with **simplified Newton correction** $\Delta \bar{x}^{(k)} := DF(x^{(k-1)})^{-1}F(x^{(k)})$.

Affine invariant: does not change when
 $F \rightarrow AF, A \in \mathbb{R}^{n,n}$ regular

C++11 code 8.5.3.6: Generic Newton iteration with termination criterion (8.5.3.5)

```

2 template <typename FuncType, typename JacType, typename VecType>
3 void newton_stc(const FuncType &F, const JacType &DF, VecType &x, double rtol,
4               double atol) {
5     using scalar_t = typename VecType::Scalar;
6     scalar_t sn;
7     do {
8         auto jacfac = DF(x).lu(); // LU-factorize Jacobian ]  $\rightarrow O(n^3)$ 
9         x -= jacfac.solve(F(x)); // Compute next iterate  $\rightarrow O(n^2)$ 
10        // Compute norm of simplified Newton correction
11        sn = jacfac.solve(F(x)).norm();  $\rightarrow O(n^2)$ 
12    }
13    // Termination based on simplified Newton correction
14    while ((sn > rtol * x.norm()) && (sn > atol));
15 }

```

▷ Review questions 8.5.3.9

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