Serie 10

Numerische Mathematik SS06

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1) $f(x) = \dfrac{1}{1+\cos^2(x)}$

$$f'(x) = -\frac{\left(1+\cos^2(x)\right)'}{\left(1+\cos^2(x)\right)^2}$$

$\longrightarrow$ Matlab : $h := 10^{-6}$ ist am besten

2) Explizites Eulerverfahren

$$x_{n+1} = x_n + h\, f(t_n, x_n) \qquad \text{für} \qquad \dot{x} = f(t,x)$$

a) $f(t,x) = -\frac{1}{2}x$, $f(t_n, x_n) = -\frac{1}{2}x_n$

exakte Lösung : $x(t) = e^{-t/2}$

b) $f(t,x) = t^2 + x^2$, $f(t_n, x_n) = t_n^2 + x_n^2$

$\longrightarrow$ Matlab

3a)
$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

$c_2 = 1/2$
$a_{21} = 1/2$
$b_1 = 0$
$b_2 = 1$

$$\left.\begin{aligned} k_1 &= f(t,x) \\ k_2 &= f\left(t + \frac{h}{2}, x + \frac{h}{2}\right) \end{aligned}\right\} \text{2 Stufen}$$

b)
$$\begin{aligned} x_{n+1} &= x_n + h\, f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\right) \\ &= x_n + h\, f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} f(t_n, x_n)\right) \end{aligned}$$

Taylorreihe in 2 Variablen:

$$g(u+h_1, v+h_2) = g(u,v)$$
$$+ \frac{1}{1!}\left(h_1 \frac{\partial g}{\partial u}(u,v) + h_2 \frac{\partial g}{\partial v}(u,v)\right)$$
$$+ \frac{1}{2!}\left(h_1^2 \frac{\partial^2 g}{\partial u^2}(u,v) + 2h_1 h_2 \frac{\partial^2 g}{\partial u \partial v}(u,v) + h_2^2 \frac{\partial^2 g}{\partial v^2}\right.$$
$$+ O(h_1^3, h_2^3)$$

$j=0,\ t_j = t_0 = 0,\ x_j = x_0$

$$x_1 = x_0 + h f\left(\frac{h}{2}, x_0 + \frac{h}{2} f(0,x_0)\right)$$
$$= x_0 + h\left[f(0,x_0) + \frac{h}{2}\frac{\partial f}{\partial t}(0,x_0) + \frac{h}{2} f(0,x_0)\frac{\partial f}{\partial x}(0,x_0)\right.$$
$$+ \frac{1}{2}\left(\frac{h^2}{4}\frac{\partial^2}{\partial t^2} f(0,x_0) + 2\frac{h}{2}\frac{h}{2} f(0,x_0)\frac{\partial^2}{\partial x \partial t} f(0,x_0) + \frac{h^2}{4} f(0,x_0)\right.$$
$$\left. + O(h^3)\right]$$
$$= x_0 + h f(0,x_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(0,x_0) + f(0,x_0)\frac{\partial f}{\partial x}(0,x_0)\right)$$
$$+ \frac{h^3}{8}\left(\frac{\partial^2}{\partial t^2} f(0,x_0) + 2\frac{\partial^2}{\partial x \partial t} f(0,x_0)\, f(0,x_0) + f(0,x_0)^2 \frac{\partial^2}{\partial x^2} f(0,x_0)\right)$$
$$+ O(h^4)$$

Andererseits hat man aber (Taylorreihe in 1 Variable)

$$x(h) = x(0) + h\dot{x}(0) + \frac{h^2}{2}\ddot{x}(0) + \frac{h^3}{6}\dddot{x}(0) + O(h^4)$$

$$\dot{x} = f(t, x(t))$$
$$\ddot{x} = \frac{d}{dt} f(t,x(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = f_t + f_x \dot{x} = f_t + f_x f$$
$$\dddot{x} = \frac{d}{dt}(f_t + f f_x) = f_{tt} + f_{tx} f + (f_t + f f_x) f_x + f(f_{xt} + f_{xx} f$$
$$= f_{tt} + 2 f f_{tx} + f_t f_x + f f_x^2 + f^2 f_{xx}$$

$$\Rightarrow x(h) = x(0) + h f\Big|_{t=0} + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right)\Big|_{t=0}$$
$$+ \frac{h^3}{6}\left(\frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial t \partial x} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} + f\left(\frac{\partial f}{\partial x}\right)^2 + f^2\frac{\partial^2 f}{\partial x^2}\right)\Big|_{t=0}$$

$\Rightarrow |x_1 - x(h)| = O(h^3)$ Also Ordnung 2.

4) $\longrightarrow$ Matlab