Lecture 1: Basic mathematical concepts

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Basic mathematical concepts

- Fourier transform;
- Kramers-Kronig relations;
- Singular value decomposition;
- Spherical mean Radon transform;
- Regularization of ill-posed problems.

Basic mathematical concepts

- Fourier transform:
$$\mathcal{F}[f](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx \,.$$
- Inverse Fourier transform:
$$\mathcal{F}^{-1}[f](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)\, dx \,.$$
- Fourier transform of a Gaussian function:
$$\mathcal{F}[e^{-|x|^2/2}](\xi) = e^{-|\xi|^2/2}, \quad \xi \in \mathbb{R}^d \,.$$
- Time reversal operation in the time domain $\Leftrightarrow$ complex conjugation in the frequency domain:
$$\mathcal{F}[f(-t)](\omega) = \overline{\mathcal{F}[f](\omega)}$$
- Cross correlation of two signals involves a product of the two Fourier transforms in the frequency domain, one of the transform being complex conjugated:
$$\mathcal{F}\Big[\int f(s)g(t+s)ds\Big](\omega) = \sqrt{2\pi}\,\overline{\mathcal{F}[f](\omega)}\mathcal{F}[g](\omega)\,.$$

Basic mathematical concepts

- f: band-limited with bandwidth K if $\mathcal{F}[f]$ vanishes outside $|\xi| \leq K$.
- Shannon's sampling theorem:
  - $f \in L^2(\mathbb{R})$: band-limited with bandwidth K; $0 < \Delta x \leq \pi/K$.
  - f: uniquely determined by the values $f(l\Delta x), l \in \mathbb{Z}$.
  - Smallest detail represented by f: of size $2\pi/K$.
  - Reconstruction formula:

$$f(x) = \sum_{l \in \mathbb{Z}} f\left(\frac{l\pi}{K}\right) \frac{\sin(Kx - l\pi)}{Kx - l\pi} .$$

Basic mathematical concepts

- Causal linear systems, with input $H(t)$, output $S(t)$, and transfer function $f(t)$:
$$S(t) = \int_{-\infty}^{t} f(t-s)H(s)\,ds \;.$$
- $f \in L^2(\mathbb{R}^+)$; $F(\omega) = \mathcal{F}[f](\omega)$.
- Paley-Wiener theorem $\Rightarrow$ $F(\omega)$: analytic for $\omega = \xi + i\eta \in \mathbb{C}^+$ and satisfies
$$\sup_{\eta>0} \int_{-\infty}^{+\infty} |F(\xi+i\eta)|^2\,d\xi = \int_{-\infty}^{+\infty} |F(\xi)|^2\,d\xi < +\infty \;.$$
- Hardy functions ($F \in \mathcal{H}^2(\mathbb{R})$).
- Converse: all Hardy functions may be obtained as Fourier transforms of L^2-functions supported on $\mathbb{R}^+$.
- Real and imaginary parts of Hardy functions obey the Kramers-Kronig relations.

Basic mathematical concepts

- Kramers-Kronig relations:
  - $F(\omega) \in \mathcal{H}^2(\mathbb{R})$,

$$\Re F(\omega) = -\frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{\Im F(\omega')}{\omega - \omega'}\, d\omega' = -\mathcal{H}[\Im F(\omega)]\,,$$

$$\Im F(\omega) = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{\Re F(\omega')}{\omega - \omega'}\, d\omega' = \mathcal{H}[\Re F(\omega)]\,.$$

  - Hilbert transform $\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$\mathcal{H}[G(\omega)] = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{G(\omega')}{\omega - \omega'}\, d\omega'\,.$$

Basic mathematical concepts

- $f(t)$: real-valued $\Rightarrow$ real and imaginary parts of its Fourier transform $F(\omega)$: respectively even and odd.
- Kramers-Kronig relations for real-valued functions $\Rightarrow$

$$\Re F(\omega) = \frac{2}{\pi} \text{p.v.} \int_0^{+\infty} \frac{\omega' \Im F(\omega')}{(\omega')^2 - \omega^2}\, d\omega' \ ;$$

$$\Im F(\omega) = -\frac{2\omega}{\pi} \text{p.v.} \int_0^{+\infty} \frac{\Re F(\omega')}{(\omega')^2 - \omega^2}\, d\omega' \ .$$

Basic mathematical concepts

- Singular value decomposition:
  - A: bounded linear operator from a separable Hilbert space H into a separable Hilbert space K.
  - Singular value decomposition (SVD) of A:

$$Af = \sum_l \sigma_l \, (f_l, f) \, g_l \; ;$$

  - $(f_l), (g_l)$: orthonormal systems in H, K, respectively;
  - σ_l: nonnegative numbers, singular values of A.

Basic mathematical concepts

- A^*: adjoint of A
$$A^* g = \sum_l \sigma_l \, (g_l, g) \, f_l \; .$$
- $A^* A f = \sum_l \sigma_l^2 \, (f_l, f) \, f_l$ and $AA^* g = \sum_l \sigma_l^2 \, (g_l, g) \, g_l$: self-adjoint operators in H, K, respectively.
- Spectrum of $A^* A, AA^*$ consists of the eigenvalues σ_l^2 and possibly the eigenvalue 0, whose multiplicity may be infinite.
- Moore-Penrose generalized inverse: least-squares solution to $Af = g$ of minimum norm,
$$A^+ g = \sum_l \sigma_l^{-1} \, (g_l, g) \, f_l \; ;$$
- Sum: over the indices l s.t. $\sigma_l > 0$.

Basic mathematical concepts

- Spherical mean Radon transform:
  - Ω: bounded open set of $\mathbb{R}^d$.
  - Spherical mean Radon transform $\mathcal{R} : \mathcal{C}^0(\mathbb{R}^d) \to \mathcal{C}^0(\partial\Omega \times \mathbb{R}^+)$ with centers on $\partial\Omega$ for $f \in \mathcal{C}^0(\mathbb{R}^d)$:

$$\mathcal{R}[f](x,s) = \frac{1}{\omega_d} \int_S f(x + s\xi)\, d\sigma(\xi), \quad (x,s) \in \partial\Omega \times \mathbb{R}^+ ;$$

  - S: unit sphere in $\mathbb{R}^d$; ω_d its area.

Basic mathematical concepts

- B: unit ball of center 0 and radius 1 ($\partial B = S$);
- $\mathcal{R} : \mathcal{C}_0^\infty(B) \to \mathcal{C}^\infty(S \times \mathbb{R}^+)$;
- Explicit inversion formulas
  - For $d = 3$:

$$f(x) = \frac{1}{2\pi} \nabla \cdot \int_S y \frac{\frac{\partial}{\partial s}(s\mathcal{R}[f])(y, |x-y|)}{|x-y|} \, d\sigma(y) \, .$$

  - For $d = 2$:

$$f(x) = \frac{1}{2\pi} \int_S \int_0^2 \left[\frac{d}{ds} s \frac{d}{ds} \mathcal{R}[f] \right] (y, s) \ln |s^2 - |y-x|^2| \, ds \, d\sigma(y) \, .$$

Basic mathematical concepts

- Relation between the spherical mean Radon transform and the wave equation.
- $y \in \mathbb{R}^3$; $U_y(x,t) := \dfrac{\delta_0(t - |x-y|)}{4\pi|x-y|}$ for $x \neq y$.
- U_y: retarded fundamental solution to the wave equation

$$(\partial_t^2 - \Delta)U_y(x,t) = \delta_y(x)\delta_0(t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}\,.$$

- U_y satisfies: $U_y(x,t) = \partial_t U_y(x,t) = 0$ for $x \neq y$ and $t < 0$.
- U_y corresponds to a spherical wave generated at the source point y and propagating at speed 1.
- In $2D$, retarded fundamental solution to the wave equation

$$U_y(x,t) := \frac{H(t - |x-y|)}{2\pi\sqrt{t^2 - |x-y|^2}} \quad \text{for } |x-y| \neq t\,;$$

- H: Heaviside step function.

Basic mathematical concepts

- Wave equation in $\mathbb{R}^d$, $d = 2, 3$,

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2}(x,t) - \Delta p(x,t) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+ \,, \\ p(x,0) = p_0(x) \quad \text{and} \quad \dfrac{\partial p}{\partial t}(x,0) = 0 \,. \end{cases}$$

- Support of $p_0 \in \mathcal{C}^0(\mathbb{R}^d)$: contained in a bounded set Ω of $\mathbb{R}^d$.
- Representation of p:

$$p(x,t) = \int_\Omega \partial_t U_y(x,t) p_0(y)\, dy, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \,.$$

Basic mathematical concepts

- Kirchhoff formulas:

$$p(x,t) = \begin{cases} \partial_t \displaystyle\int_0^t \frac{s\mathcal{R}[p_0](x,s)}{\sqrt{t^2 - s^2}}\, ds\,, & d = 2\,,\\ \partial_t (t\mathcal{R}[p_0])(x,t), & d = 3\,. \end{cases}$$

- Ω: unit disk ($\partial\Omega = S$),

$$p_0(x) = \mathcal{R}^*\mathcal{B}\mathcal{R}[p_0](x)\,;$$

- $\mathcal{R}^*$: adjoint of $\mathcal{R}$,

$$\mathcal{R}^*[g](x) = \frac{1}{2\pi}\int_S \frac{g(y, |x-y|)}{|x-y|}\, d\sigma(y)\,;$$

- Filter $\mathcal{B}$: for $g : S \times \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{B}[g](x,t) = \int_0^2 \frac{\partial^2 g}{\partial s^2}(x,s) \ln(|s^2 - t^2|)\, ds.$$

Basic mathematical concepts

- Regularization of ill-posed problems:
  - Ill-posed problem:
    - It may not be solvable (in the strict sense) at all.
    - The solution, if exists, may not be unique.
    - The solution may not depend continuously on the data.
  - Regularization methods:
    - Truncated SVD;
    - Tikhonov-Phillips regularization;
    - Truncated iterative methods.

Basic mathematical concepts

- Deconvolution problem:
  - h: Gaussian convolution kernel, $h(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$;

    $\mathcal{F}[h](\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}$.
  - $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$(Af)(x) := \int_{-\infty}^{+\infty} h(x-y) f(y)\, dy \, .$$

  - A: compact.
  - $\mathcal{F}[Af] = \mathcal{F}[h \star f] = \mathcal{F}[h]\mathcal{F}[f] \Rightarrow A$: injective.
  - Solution to $Af = g$:

$$f(x) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}(g)}{\mathcal{F}(h)}\right](x), \quad x \in \mathbb{R} \, .$$

  - Formula: not well defined for general $g \in L^2(\mathbb{R}) \Leftarrow 1/\mathcal{F}[h]$ grows as $e^{\xi^2/2}$.

Basic mathematical concepts

- A: bounded linear operator from a Hilbert space H into a Hilbert space K.
- Consider the problem of solving $Af = g$ for f:
  - g may not be in the range of A;
  - A may not be injective;
  - A^{-1} may not be continuous.
- SVD of A:

$$Af = \sum_l \sigma_l \, (f_l, f) \, g_l.$$

- Truncated SVD:

$$T_\gamma g = \sum_{\sigma_l \geq \gamma} \sigma_l^{-1} \, (g_l, g) \, f_l.$$

- $|| T_\gamma || \leq 1/\gamma$.

Basic mathematical concepts

- Tikhonov-Phillips regularization:
  - Linear problems:

$$T_\gamma = (A^* A + \gamma I)^{-1} A^* \, .$$

  - Equivalently, $f_\gamma = T_\gamma g$: defined by minimizing

$$||Af - g||^2 + \gamma ||f||^2 \, .$$

  - γ: regularization parameter.
  - In terms of the SVD of A:

$$T_\gamma g = \sum_l F_\gamma(\sigma_l) \sigma_l^{-1} \left(g_l, g \right) f_l \, ;$$

  - $F_\gamma(\sigma) = \sigma^2 / (\sigma^2 + \gamma)$.

Basic mathematical concepts

- Tikhonov-Phillips regularization:
  - Nonlinear problems:
  - $A : H \to K$ nonlinear mapping.
  - $f_\gamma = T_\gamma g$: defined by minimizing

$$||A(f) - g||^2 + \gamma G(f);$$

  - $G : H \to \mathbb{R}$: nonnegative functional ($G(f) = ||f||^2$).
  - A: Fréchet differentiable.
  - R_{f_0} is the Fréchet derivative of A at f_0:

$$A(f_0 + h) = A(f_0) + R_{f_0} h + o(||h||) .$$

  - Linearization of A around a given point $f_0 \Rightarrow$ minimizer (around f_0):

$$f = (R_{f_0}^* R_{f_0} + \gamma I)^{-1} R_{f_0}^* \Big(g - A(f_0) + R_{f_0} f_0 \Big) .$$

Basic mathematical concepts

- Regularization by truncated iterative methods
- Drawback of the Thikhonov-Phillips regularization: inversion of the regularized normal operator $A^*A + \gamma I$ may be costly in practice.
- Linear Landweber iteration: iterative technique

$$f^0 = 0, \quad f^{k+1} = f^k + \eta A^* \big(g - Af^k\big), \quad k \geq 0 \,,$$

for some $\eta > 0$ small.

- By induction,

$$f^{k+1} = (I - \eta A^* A) f^k + \eta A^* g, \quad k \geq 0 \,,$$

- $\Rightarrow f^k = T_\gamma g$, with $\gamma = 1/k, k \geq 1$, and

$$T_\gamma g = \eta \sum_{l=0}^{1/\gamma - 1} (I - \eta A^* A)^l A^* g \,.$$

Basic mathematical concepts

- $q \leq +\infty$: number of singular values of A;
- σ_l: singular values arranged in a decreasing sequence and g_l and f_l be respectively the associated left and right singular vectors.
- $\eta \approx \sigma_1^{-2}$: good choice $\Leftarrow$

$$\eta \sum_{l=0}^{1/\gamma - 1} (I - \eta A^* A)^l A^* g = \sum_{l=1}^{q} \frac{1}{\sigma_l} (1 - (1 - \eta \sigma_l^2)^{1/\gamma}) (g_l, g) f, g_l) \, .$$

Basic mathematical concepts

- Nonlinear Landweber iterations
- H: Hilbert space and K: closed and convex subset of H.
- $F : K \to H$: Fréchet differentiable.
- Given $y_* \in H$,

$$F(x_*) = y_*, \quad x_* \in K .$$

- Minimize for $x \in K$

$$J(x) = \frac{1}{2} \|F(x) - y_*\|^2 .$$

- F: Fréchet differentiable $\Rightarrow$ J: Fréchet differentiable

$$F(x+h) = F(x) + dF(x)h + o(||h||);$$

$$dJ(x)h = \big(dF(x)h, F(x) - y_*\big) = \big(h, dF(x)^* (F(x) - y_*)\big).$$

Basic mathematical concepts

- Nonlinear Landweber iteration:

$$x^{(n+1)} = Tx^{(n)} - \eta dF(Tx^{(n)})^*(F(Tx^{(n)}) - y_*) \, .$$

- T: Hilbert projection of H onto K

$$T : H \ni x \mapsto \mathrm{argmin}\{\|x - a\} : a \in K\};$$

- η: small positive number.
- Assume that the Fréchet derivative dF: Lipschitz continuous and that for all $x \in K$,

$$\|dF(x)\|_{H^*} \geq c \, ,$$

for some positive constant c. Then the nonlinear Landweber iteration converges to x_* provided that $x^{(0)}$: "good" initial guess for x_* and η: sufficiently small.

Basic mathematical concepts

- Proof:
  - dF: Lipschitz continuous $\Rightarrow$ for all x such that $\|x - x_*\|$: small

$$\|F(x) - F(x_*) - dF(x)(x - x_*)\| \leq C\|x - x_*\|^2$$
$$\leq C\|x - x_*\|\|F(x) - F(x_*)\| \leq \mu\|F(x) - F(x_*)\|$$

for small constant μ.

  - For all $n \geq 1$, F_n: nth error quantity $F(Tx^{(n)}) - y_*$,

$$\|x^{(n+1)} - x_*\|^2 - \|x^{(n)} - x_*\|^2$$
$$\leq 2\langle x^{(n+1)} - Tx^{(n)}, Tx^{(n)} - x_*\rangle + \|x^{(n+1)} - Tx^{(n)}\|^2$$
$$\leq \langle F_n, 2\eta F_n - 2\eta dF(Tx^{(n)})(Tx^{(n)} - x_*)\rangle - \eta\|F_n\|^2$$
$$+\langle \eta F_n, (-I + \eta dF(Tx^{(n)}) dF(Tx^{(n)})^*)) F_n\rangle$$
$$\leq \eta(2\mu - 1)\|F_n\|^2 .$$

Basic mathematical concepts

- $\Rightarrow$
$$\|x^{(n+1)} - x_*\|^2 + \eta(1-2\mu)\|F_n\|^2 \leq \|x^{(n)} - x_*\|^2\,,$$
- $\Rightarrow$
$$\sum_{n=1}^{\infty} \|FT(x^{(n)}) - y_*\|^2 \leq \frac{\|x^{(0)} - x_*\|^2}{\eta(1-2\mu)}\,.$$
- Mean value theorem $\Rightarrow$
$$c\| Tx^{(n)} - x_*\| \leq \|dF(\tilde{x}^{(n)})(Tx^{(n)} - x_*)\| = \|F(Tx^{(n)}) - F(x_*)\| \to 0$$
for some $\tilde{x}^{(n)} = tTx^{(n)} + (1-t)x_*,\ t \in (0,1)$.