Lecture 10: Viscoelastic modulus reconstruction and full-field OCT elastography

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Elastography

- Elastography: quantitative visualization of the mechanical properties of human tissues by using the relation between the wave propagation velocity and the mechanical properties of the tissues.
- Mechanical properties of tissue include the shear modulus, shear viscosity, and compression modulus.
- Quantification of the tissue shear modulus in vivo can provide evidence of the manifestation of tissue diseases.
- Image reconstruction methods for tissue viscoelasticity imaging.
- Recover the distribution of the shear modulus (μ) and shear viscosity (η) from the internal measurement of the time-harmonic mechanical displacement field u produced by the application of an external time harmonic excitation at frequency $\omega/2\pi$ in the range $50 \sim 200$Hz through the surface of the subject.
- Modeling soft tissue as being linearly viscoelastic and nearly incompressible.

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- Displacement: $\Re(u(x)e^{i\omega t})$,

$$\nabla \cdot \Big((\mu + i\omega\eta)(\nabla u + \nabla u^T)\Big) + \nabla((\lambda + i\omega\eta_\lambda)\nabla \cdot u) + \rho\omega^2 u = 0\,;$$

  ρ: density of the medium, ∇u^T: transpose of ∇u, λ: compression modulus and η_λ: compression viscosity.

- Algebraic inversion method: For any non-zero constant vector a,

$$\mu + i\omega\eta = -\frac{\rho\omega^2(a \cdot u)}{\nabla \cdot \nabla(a \cdot u)}\,.$$

- Strong assumptions of $\nabla(\mu + i\omega\eta) \approx 0$ (local homogeneity) and $(\lambda + i\omega\eta_\lambda)\nabla \cdot u \approx 0$ (negligible pressure).
- Algebraic formula: ignores reflection effects of the propagating wave due to abrupt changes of $\mu + i\omega\eta$, so that the method cannot measure any change of $\mu + i\omega\eta$ in the direction of a.

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- Boundary conditions: Γ_D and Γ_N s.t. $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$,

$$\begin{aligned} u = g \qquad & \text{on } \Gamma_D\,, \\ 2(\mu + i\omega\eta)\nabla^s u\,\nu + (\lambda + i\omega\eta_\lambda)(\nabla\cdot u)\nu = 0 \qquad & \text{on } \Gamma_N\,. \end{aligned}$$

- Soft tissues: nearly incompressible.
- $\lambda \approx \infty \Rightarrow \nabla\cdot u \approx 0$.
- Internal pressure $p = \lim_{\lambda\to+\infty} \lambda\nabla\cdot u$.
- Quasi-incompressible viscoelasticity model:

$$\begin{cases} 2\nabla\cdot((\mu + i\omega\eta)\nabla^s u) + \nabla p + \rho\omega^2 u = 0 & \text{in } \Omega\,, \\ \nabla\cdot u = 0 & \text{in } \Omega\,, \\ u = g & \text{on } \Gamma_D\,, \\ 2(\mu + i\omega\eta)\nabla^s u\,\nu + p\nu = 0 & \text{on } \Gamma_N\,. \end{cases}$$

- If $\Gamma_D = \partial\Omega$ ($\Gamma_N = \emptyset$), then g should satisfy the compatibility condition $\int_{\partial\Omega} g\cdot\nu\, ds = 0$.
- $u^{(m)} = u^{(m)}[\mu_*, \eta_*]$: displacement measured in Ω; μ_* and η_*: true distributions of shear elasticity and viscosity.
- Inverse problem: reconstruct the distribution of μ and η from $u^{(m)}$.

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- For a fixed $\epsilon > 0$; $\Omega' := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \epsilon\}$ and $\mathcal{E} := \Omega \backslash \overline{\Omega'}$.
- μ and $\eta \in \widetilde{S} := \{(\mu_0, \eta_0) + (\phi_1, \phi_2) \mid (\phi_1, \phi_2) \in S\}$; μ_0 and η_0: positive constants.
- $\tilde{S} = (\mu_0, \eta_0) + S$:

$$S := \Big\{(\phi_1, \phi_2) \in H_0^2(\Omega) \times H_0^2(\Omega) : \ c_1 < \phi_1 + \mu_0 < c_2, \\ c_1 < \phi_2 + \eta_0 < c_2, \|\phi_j\|_{W^{2,2}(\Omega)} \leq c_3, \ \text{supp } \phi_j \subset \Omega' \text{ for } j = 1, 2\Big\};$$

c_1, c_2, c_3: positive constants.

Elastography

- Optimal control algorithm.
- Discrepancy functional:

$$J[\mu,\eta] = \frac{1}{2}\int_\Omega \left|u[\mu,\eta] - u^{(m)}\right|^2 dx\,.$$

- Fréchet derivatives of $J[\mu,\eta]$ with respect to μ and η.
- Assume that δ_μ and δ_η: small perturbations of μ and η, respectively, by regarding $\frac{\delta\mu + i\omega\delta_\eta}{\mu + i\omega\eta} \approx 0$.
- $u_0 := u[\mu,\eta]$, $p_0 :=$ the pressure corresponding to u_0 and $p_0 + p_1 :=$ the pressure corresponding to $u[\mu+\delta_\mu, \eta+\delta_\eta]$;

$$\delta u := u[\mu+\delta_\mu, \eta+\delta_\eta] - u_0\,.$$

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$$\begin{aligned}2\nabla\cdot((\mu+i\omega\eta)\nabla^s\delta u) + \nabla p_1 + \rho\omega^2\delta u = &-2\nabla\cdot((\delta_\mu + i\omega\delta_\eta)\nabla^s u_0)\\ &-2\nabla\cdot((\delta_\mu + i\omega\delta_\eta)\nabla^s\delta u) \quad \text{in} \quad \Omega\,.\end{aligned}$$

Elastography

- u_1: solution of

$$\begin{cases} 2\nabla \cdot ((\mu + i\omega\eta)\nabla^s u_1) + \nabla p_1 + \rho\omega^2 u_1 = \\ \qquad\qquad -2\nabla \cdot ((\delta_\mu + i\omega\delta_\eta)\nabla^s u_0) & \text{in } \Omega, \\ \nabla \cdot u_1 = 0 & \text{in } \Omega\,, \\ u_1 = 0 & \text{on } \Gamma_D\,, \\ 2(\mu + i\omega\eta)\nabla^s u_1 \nu + p_1 \nu = 0 & \text{on } \Gamma_N\,. \end{cases}$$

- For $(\delta\mu + \mu, \delta\eta + \eta) \in \widetilde{S}$,

$$\Re \int_\Omega u_1 \overline{(u_0 - u^{(m)})}\, dx = \Re \int_\Omega 2(\delta_\mu + i\omega\delta_\eta)\nabla^s u_0 : \nabla^s \overline{v}\, dx\,.$$

- Fréchet derivatives of $J[\mu, \eta]$ with respect to μ and η:

$$\frac{\partial}{\partial\mu} J[\mu, \eta] = \Re\,[2\nabla^s u_0 : \nabla^s \bar{v}]\,,\ \frac{\partial}{\partial\eta} J[\mu, \eta] = \Re\,[2(i\omega\nabla^s u_0) : \nabla^s \bar{v}]\ ;$$

Elastography

- v solution of the adjoint problem:

$$\begin{cases} 2\nabla \cdot ((\mu - i\omega\eta)\nabla^s v) + \nabla q + \rho\omega^2 v = (u_0 - u^{(m)}) & \text{in } \Omega\,, \\ \nabla \cdot v = 0 & \text{in } \Omega\,, \\ v = 0 & \text{on } \Gamma_D\,, \\ 2(\mu - i\omega\eta)\nabla^s v\, \nu + q\nu = 0 & \text{on } \Gamma_N\,. \end{cases}$$

- $J[\mu, \eta]$: Fréchet differentiable for $(\mu, \eta) \in \widetilde{S}$.
- As $\delta_\mu, \delta_\eta \to 0$,

$$\left| J[\mu + \delta_\mu, \eta + \delta_\eta] - J[\mu, \eta] - \Re \int_\Omega u_1 (\overline{u_0 - u^{(m)}}) dx \right| = O\left((||\delta_\mu||_{W^{2,2}(\Omega)} + ||\delta_\eta||_{W^{2,2}(\Omega)})^2 \right)\,.$$

Elastography

- Gradient descent scheme:
  - Let $n = 0$. Start with an initial guess of shear modulus $\mu^{(0)}$ and shear viscosity $\eta^{(0)}$.
  - For $n = 0, 1, \ldots$, compute $u_0^{(n)}$ by solving the forward problem with μ and η replaced by $\mu^{(n)}$ and $\eta^{(n)}$, respectively. Compute $v^{(n)}$ by solving the adjoint problem with μ, η, u_0 replaced by $\mu^{(n)}, \eta^{(n)}, u_0^{(n)}$, respectively.
  - For $n = 0, 1, \ldots$, compute the Fréchet derivatives $\frac{\partial J}{\partial \mu}[\mu^{(n)}, \eta^{(n)}]$ and $\frac{\partial J}{\partial \eta}[\mu^{(n)}, \eta^{(n)}]$.
  - Update μ and η as follows:

$$\begin{cases} \mu^{(n+1)} & = \mu^{(n)} - \delta \frac{\partial J}{\partial \mu}[\mu^{(n)}, \eta^{(n)}] \,, \\ \eta^{(n+1)} & = \eta^{(n)} - \delta \frac{\partial J}{\partial \eta}[\mu^{(n)}, \eta^{(n)}] \,, \end{cases}$$

  δ: step size.
  - Repeat Steps $2, 3$, and 4 until $||\mu^{(n+1)} - \mu^{(n)}|| \leq \epsilon$ and $||\eta^{(n+1)} - \eta^{(n)}|| \leq \epsilon$ for a given $\epsilon > 0$.

Elastography

- Initial guess:
  - Ignore the pressure term:

$$2\nabla \cdot (\mu + i\omega\eta)\nabla^s u^\diamond + \rho\omega^2 u^\diamond = 0 \quad \text{in } \Omega ,$$

  - Helmholtz decomposition:

$$(\mu + i\omega\eta)\nabla^s u^\diamond = \nabla f + \nabla \times W \text{ with } \nabla \cdot W = 0 ,$$

  - f and W: vector and matrix, respectively.

$$\mu + i\omega\eta = \frac{\nabla f : \nabla^s \bar{u}^\diamond}{|\nabla^s u^\diamond|^2} + \frac{\nabla \times W : \nabla^s \bar{u}^\diamond}{|\nabla^s u^\diamond|^2} .$$

$$\Delta f = -\frac{1}{2}\rho\omega^2 u^\diamond \quad \text{in } \Omega .$$

$$\Delta W = \nabla \times ((\mu + i\omega\eta)\nabla^s u^\diamond) \quad \text{in } \Omega .$$

Elastography

- $\widetilde{f}$:

$$\begin{cases} \Delta \widetilde{f} = -\frac{1}{2}\rho\omega^2 u^{(m)} & \text{in } \Omega\,, \\ \nabla \widetilde{f}\,\nu = (\mu_0 + i\omega\eta_0)\nabla^s u^{(m)}\,\nu & \text{on } \partial\Omega\,. \end{cases}$$

- W_1:

$$\begin{cases} \Delta W_1 = \nabla \times \left(\frac{\nabla \widetilde{f} : \nabla^s \bar{u}^{(m)}}{|\nabla^s u^{(m)}|^2} \nabla^s u^{(m)}\right) & \text{in } \Omega\,, \\ W_1 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

- W_2:

$$\begin{cases} \Delta W_2 = \nabla \times \left(-\frac{\rho\omega^2 (a\cdot u^{(m)})}{\nabla\cdot\nabla(a\cdot u^{(m)})} \nabla^s u^{(m)}\right) & \text{in } \Omega\,, \\ W_2 = 0 & \text{on } \partial\Omega\,, \end{cases}$$

a: any nonzero vector.

Elastography

- Initial guess formula:

$$\mu^{(0)} + i\omega\eta^{(0)} = \frac{\nabla \widetilde{f} : \nabla^s \bar{u}^{(m)}}{|\nabla^s u^{(m)}|^2} + \frac{\nabla \times (W_1 + W_2) : \nabla^s \bar{u}^{(m)}}{2|\nabla^s u^{(m)}|^2} \, .$$

Elastography

- Local reconstruction:
  - $\Omega_{\text{loc}} \Subset \Omega$; Localized minimization problem:

$$J_{\text{loc}}[\mu, \eta] = \frac{1}{2} \int_{\Omega_{\text{loc}}} |u_{\text{loc}}[\mu, \eta] - u^{(m)}|^2 dx;$$

  - $u_{\text{loc}}[\mu, \eta]$:

$$\begin{cases} 2\nabla \cdot ((\mu + i\omega\eta)\nabla^s u) + \nabla p + \rho\omega^2 u = 0 & \text{in } \Omega_{\text{loc}}, \\ \nabla \cdot u = 0 & \text{in } \Omega_{\text{loc}}, \\ u = u^{(m)} & \text{on } \partial\Omega_{\text{loc}}. \end{cases}$$

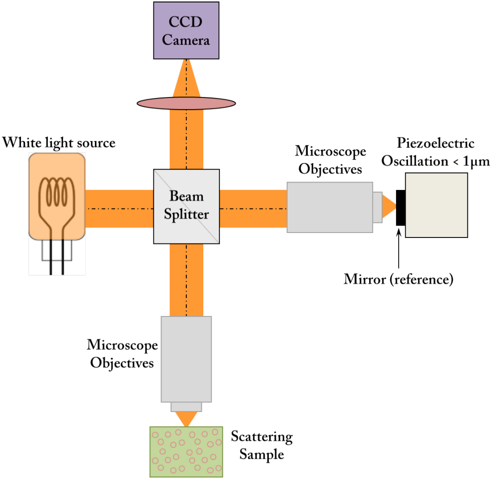
  - Adjoint problem:

$$\begin{cases} 2\nabla \cdot ((\mu - i\omega\eta)\nabla^s v) + \nabla q + \rho\omega^2 v = u_{\text{loc}} - u^{(m)} & \text{in } \Omega_{\text{loc}}, \\ \nabla \cdot v = 0 & \text{in } \Omega_{\text{loc}}, \\ v = 0 & \text{on } \partial\Omega_{\text{loc}}. \end{cases}$$

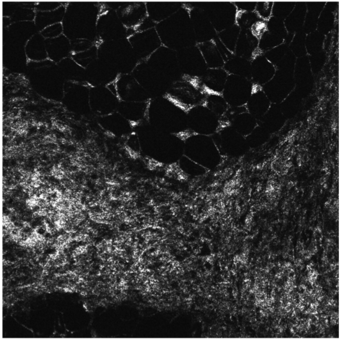
Full-field optical coherence elastography

- Full-field optical coherence tomography (OCT): optical image with sub-cellular resolution.
- Apply a load on the sample.
- OCTE: Use a set of optical images before and after mechanical solicitation to reconstruct the shear modulus distribution inside the sample.
- Map of mechanical properties: added as a supplementary contrast mechanism to enhance specificity.

Full-field optical coherence elastography



Full-field optical coherence elastography



Full-field optical coherence elastography

- Reconstruct the shear modulus μ from ε and ε_u.
- $\varepsilon(x) = \varepsilon_u\,(x + u(x))\,;$
- Displacement field u:

$$\left\{\begin{aligned} \nabla\cdot\Big(\mu(\nabla u + \nabla u^T)\Big) + \nabla p = 0 \quad &\text{in}\ \ \Omega,\\ \nabla\cdot u = 0 \quad &\text{in}\ \ \Omega,\\ u = f \quad &\text{on}\ \ \partial\Omega. \end{aligned}\right.$$

Full-field optical coherence elastography

- $\mathrm{BV}(\Omega)$: subspace of $L^1(\Omega)$ of all the functions f whose weak derivative Df: a finite Radon measure:
- f s.t.

$$\int_\Omega f \nabla \cdot F \leq C \sup_{x \in \Omega} |F|, \quad \forall\, F \in \mathcal{C}_0^1(\Omega)^d$$

for some positive constant C with $\mathcal{C}_0^1(\Omega)$: set of compactly supported $\mathcal{C}^1$ functions.

Full-field optical coherence elastography

- Derivative of a function $f \in \mathrm{BV}(\Omega)$ can be decomposed as

$$Df = \nabla f \mathcal{H}^d + [f]\nu_s \mathcal{H}_S^{d-1} + D_c f;$$

- $\mathcal{H}^d$: Lebesgue measure on Ω, $\mathcal{H}_S^{d-1}$: surface Hausdorff measure on a rectifiable surface S, ν_S: normal vector defined a.e. on S;
- $\nabla f \in L^1(\Omega)$: smooth derivative of f, $[f] \in L^1(S, \mathcal{H}_S^{d-1})$: jump of f across S and $D_c f$: vector measure supported on a set of Hausdorff dimension less than $(d-1)$.

Full-field optical coherence elastography

- $\mathrm{SBV}(\Omega)$: subspace of $\mathrm{BV}(\Omega)$ of all the functions f satisfying $D_c f = 0$.
- For any $1 \leq p \leq +\infty$,

$$\mathrm{SBV}^p(\Omega) = \left\{ f \in \mathrm{SBV}(\Omega) \cap L^p(\Omega), \; \nabla f \in L^p(\Omega)^d \right\} .$$

Full-field optical coherence elastography

- u^*: true displacement; $\widetilde{\varepsilon}$: measured deformed optical:

$$\widetilde{\varepsilon} = \varepsilon \circ (\mathbb{I} + u^*)^{-1}.$$

- Optical image: discontinuous.
- Optimal control algorithm:

$$I(u) = \frac{1}{2} \int_\Omega |\widetilde{\varepsilon} \circ (\mathbb{I} + u) - \varepsilon|^2 \, dx.$$

- I has a nonempty subgradient.
- $\xi \in \partial I$:

$$\xi : h \mapsto \int_\Omega [\widetilde{\varepsilon}(x+u) - \varepsilon(x)] h(x) \cdot D\widetilde{\varepsilon} \circ (\mathbb{I} + u)(x) \, dx.$$

Full-field optical coherence elastography

- Nondifferentiable functional $u \mapsto I(u)$ has a nonempty subgradient if there exists ξ s.t.
$$I(u+h) - I(u) \geq (\xi, h)$$
holds for $||h||$ small enough;
- $\xi \in \partial I$ with ∂I: subgradient of I.
- In order to minimize I, it is sufficient to find one $\xi \in \partial I$.

Full-field optical coherence elastography

Initial guess:

- Detect the surface of jumps of the optical image (edge detection algorithm).
- Local recovery by linearization: $\mathrm{data} = \varepsilon - \varepsilon_u (\approx \nabla\varepsilon \cdot u)$

$$J_x(u) = \int_\Omega |\nabla\varepsilon(y) \cdot u - \mathrm{data}(y)|^2 w_\delta(|x-y|) dy.$$

- $w_\delta = \frac{1}{\delta^d} w\left(\frac{\cdot}{\delta}\right)$; w: a mollifier supported on $[-1, 1]$.
- Least-squares solution:

$$u^T = \left(\int_\Omega w_\delta(|x-y|)\nabla\varepsilon(y)\nabla\varepsilon^T(y)dy\right)^{-1} \int_{x+\delta B} \mathrm{data} w_\delta(|x-y|)\nabla\varepsilon(y)dy.$$

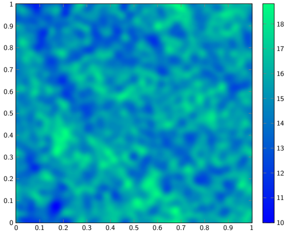
- If all vectors $\nabla\varepsilon$ in $\{y \ : \ w_\delta(|y-x|) \neq 0\}$ not collinear, then

$$\int_\Omega w_\delta(|x-y|)\nabla\varepsilon(y)\nabla\varepsilon^T(y)dy \quad \text{invertible}.$$

- Resolution $=$ variation of ε.

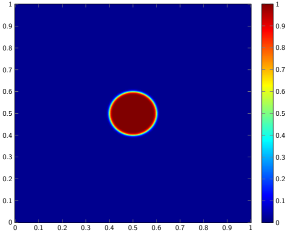
Full-field optical coherence elastography

Optical image ε of the medium:

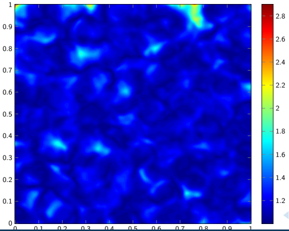


Full-field optical coherence elastography

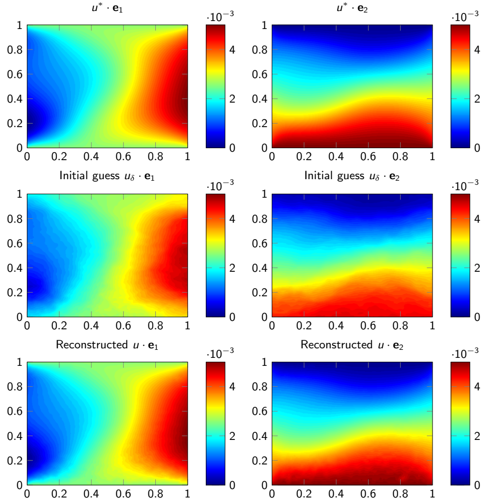
Averaging kernel w_δ:



Conditioning of the matrix $w_\delta \star \nabla\varepsilon\nabla\varepsilon^T$:

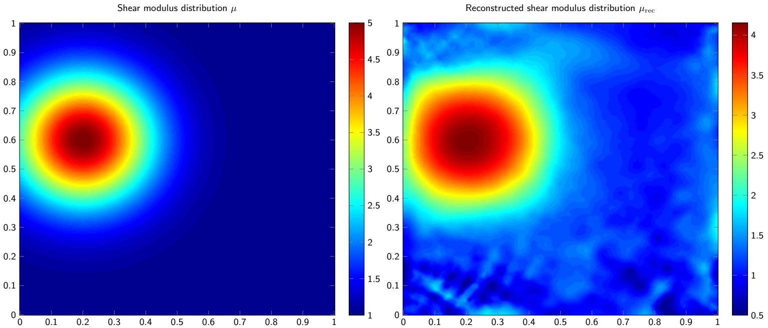


Full-field optical coherence elastography



Displacement field and its reconstruction.

Full-field optical coherence elastography



Shear modulus reconstruction.