Lecture 2: Tissues properties

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- Properties of biological tissues:
  - Electric properties;
  - Optical properties;
  - Elastic properties.
- Governing models:
  - Maxwell's equations $\Rightarrow$ conductivity equation; Helmholtz equation; diffusion approximation.
  - Lamé equation $\Rightarrow$ acoustic wave equation; modified Stokes system.

Electrical properties of tissues

- Magnetic permeability $\mu =$ free space $= 1$.
- Electrical conductivity σ: tissue's ability to transport charges;
- Electrical permittivity (dielectric constant) ε': tissue's ability to trap or to rotate molecular dipoles; determined by the polarization under an external electric field; free space electrical permittivity $= 1$.
  - σ and ε': frequency-dependent or dispersive; ω: frequency of the alternating current.
  - Capacitive effect generated by the cell membrane structure.
  - $\sigma(\omega) = \sigma_0 + \omega\varepsilon''(\omega)$; ε'': loss factor; σ_0: conductivity at very low frequencies.
  - $\varepsilon(\omega) = \varepsilon'(\omega) - i\varepsilon''$: complex permittivity.
- Electrical admittivity $\kappa = \sigma(\omega) + i\omega\varepsilon'$; macroscopic parameter; represents the electrical properties of the tissue averaged in space over many cells; can be anisotropic.

Electrical properties of tissues

- Causality $\Rightarrow$ Kramers-Krönig relations (Hilbert transform):

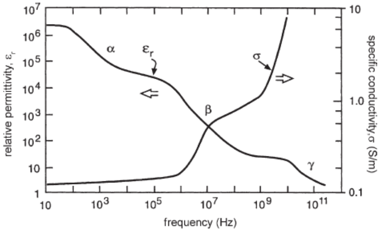
$$\varepsilon'(\omega) - \varepsilon_\infty = \frac{2}{\pi}\text{p.v.} \int_0^{+\infty} \frac{s\varepsilon''(s)}{s^2 - \omega^2} ds,$$

$$\varepsilon''(\omega) = -\frac{2\omega}{\pi}\text{p.v.} \int_0^{+\infty} \frac{\varepsilon'(s) - \varepsilon_\infty}{s^2 - \omega^2} ds;$$

- ε_∞: dielectric constant at very high frequencies.
- p.v.: principle value.

Electrical properties of tissues

- Dispersion: significant change in the dielectric properties over a frequency range.
- Relaxation mechanisms (depend on the tissue):
  - α-dispersion: low frequencies (80 Hz for muscle)
  - β-dispersion: radio frequencies (50 KHz)
  - γ-dispersion: microwave frequencies (25 GHz); σ increases with ω (dipolar reorientation of tissue water); ε' decreases.



Electrical properties of tissues

- Empirical approaches:
  - Debye model:

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + i\omega\tau}$$

  - Cole-Cole model:

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + (i\omega\tau)^\alpha}$$

  - ε_0: dielectric constant at very low frequencies; τ: relaxation time; τ and $0 < \alpha < 1$: depend on the nature of the biological material.

Electrical properties of tissues

- Maxwell's equations:

$$\begin{cases} \nabla \times E = -\dfrac{\partial H}{\partial t}, \quad \nabla \times H = J + \dfrac{\partial D}{\partial t}, \\ \nabla \cdot H = 0, \quad \nabla \cdot D = \rho. \end{cases}$$

- E: electric field; H: magnetic field; D: electric flux density; $\partial D/\partial t$: displacement current; J: ellectric current density; ρ: charge density.
- Equation of conservation of charges:

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0.$$

- Ohm's law:

$$J = \sigma_0 E \quad \text{in } \Omega \times \mathbb{R}_+.$$

- Total current density $J_{\text{tot}} = J + \partial D/\partial t = \sigma_0 E + \partial D/\partial t$.
- Causal constitutive relationship:

$$(*) \qquad D(x,t) = \int_{-\infty}^{t} \varepsilon(x, t-s) E(x,s) ds, \quad (x,t) \in \Omega \times \mathbb{R}^+.$$

Electrical properties of tissues

- Time-harmonic solutions:

$$E(x,t) = \Re E(x,\omega)e^{i\omega t}, \quad H(x,t) = \Re H(x,\omega)e^{i\omega t}.$$

- Constitutive relation: $(*)$ becomes in the frequency domain multiplicative

$$D(x,\omega) = \varepsilon(x,\omega)E(x,\omega).$$

- $\varepsilon(x,\omega) := \int_{-\infty}^{+\infty} \varepsilon(x,t)e^{-i\omega t}\, dt$ satisfies the Kramers-Kronig relations: frequency-domain expression of causality.
- Maxwell's equations:

$$\nabla \times \nabla \times E - \omega^2 \big(\varepsilon' + \frac{\sigma(\omega)}{i\omega}\big) E = 0.$$

Electrical properties of tissues

- $n^2 = \varepsilon' + \dfrac{\sigma}{i\omega}$;
- $\nabla \times \nabla \times = -\Delta + \nabla\nabla \cdot \Rightarrow$

$$\Delta E + \omega^2 n^2 E + 2\nabla\big(\frac{\nabla n}{n} \cdot E\big) = 0.$$

- Assumption: ε' and ε'' vary slowly on the scale of the wavelength $2\pi/\omega$ $\Rightarrow$
- Microwave frequencies (slow variations of ε), E_j solution to the Helmholtz equation:

$$(**) \qquad \Delta E_j + \omega^2 n^2 E_j = 0.$$

- $(**)$: scalar approximation of Maxwell's equations.
- $\omega \to 0$, $E = \nabla u$: u solution to the conductivity equation

$$(***) \qquad \nabla \cdot \big(\sigma + i\omega\varepsilon'\big)\nabla u = 0.$$

$(***)$: quasi-static approximation of Maxwell's equations.

Optical properties of tissues

- Electromagnetic fields transmitted through a biological medium with varying degrees of
  - absorption: function of the molecular composition of the tissue $\Rightarrow$ sensitive to tissue pathologies and functions.
  - reflection: occurs at tissues boundaries;
  - scattering: caused by inhomogeneities of cellular structures and particle sizes of the order of the optical wavelength.

Optical properties of tissues

- Mathematical description of light propagation: three scales
  - Maxwell's equations in random media: microscopic scale.
  - Radiative transport equation (RTE): mesoscale; high-frequency limit of the intensity $I(x, \xi, t)$ at the position x and direction ξ; characteristic scale= scattering length.
  - Diffusion approximation to the RTE: macroscale; approximation of the RTE.

Optical properties of tissues

- Assumption: medium composed of spatially uncorrelated point particles with number density ρ.
- σ_s and σ_a: scattering and absorption cross-sections of the particle;
- Absorption coefficient $\mu_a = \rho\sigma_a$; Scattering coefficient $\mu_s = \rho\sigma_s$;
- μ_a and μ_s: depend on the wavelength.
- $\Psi(x,t) = \displaystyle\int I(x,\xi,t)\,d\xi$: energy density or fluence rate:

$$\frac{1}{c}\frac{\partial\Psi}{\partial t} - \nabla\cdot\left[\frac{1}{3(1-g)(\mu_s+\mu_a)}\nabla\Psi\right] + \mu_a\Psi = 0.$$

- g: coefficient of anisotropy
  - $g = 0$: isotropic scattering;
  - $g = 1$: extreme forward scattering.
- Diffusion approximation: valid $1/\mu_s$ (scattering mean path) $\ll$ distance of propagation.

Optical properties of tissues

- Boundary condition:

$$(***{*})\qquad \Psi + l_{\rm ext}\nu\cdot\nabla\Psi = f \quad \text{on } \partial\Omega\times\mathbb{R}^+;$$

- f: source;
- $l_{\rm ext}$: extrapolation length computed from the radiative transfer theory
  - $l_{\rm ext} = 0$: absorbing boundary;
  - $l_{\rm ext} \to +\infty$: reflecting boundary.
- Time-harmonic source: $f(x,t) = \Re(f(x)e^{i\omega t}) \Rightarrow \Psi(x,t) = \Re(\Psi(x)e^{i\omega t})$:

$$-\nabla\cdot\left[\frac{1}{3(1-g)(\mu_s+\mu_a)}\nabla\Psi\right] + (\mu_a + \frac{i\omega}{c})\Psi = 0 \quad \text{in } \Omega,$$

with the boundary condition $(***{*})$.

Elastic properties of tissues

- (λ, μ): Lamé coefficients; ρ: density; $\beta := \lambda + 2\mu/d$: bulk modulus ($d$: space dimension).
- Lamé system:

$$\begin{cases} \rho \dfrac{\partial u}{\partial t^2} - \nabla \lambda \nabla \cdot u - 2\nabla \cdot \mu \nabla^s u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial n} = F & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{in } \Omega. \end{cases}$$

- $\nabla^s = (\nabla + \nabla^T)/2$; T: transpose.
- Co-normal derivative: $\frac{\partial u}{\partial n} = \lambda(\nabla \cdot u)\nu + 2\mu \nabla^s u \, \nu$.
- ν: normal to $\partial\Omega$.
- $\frac{\partial u}{\partial n}$ on $\partial\Omega$: sample/ air interface.
- Strain tensor: $\nabla^s u$; stress tensor: $\sigma(u) = \mathbb{C}\nabla^s u$; elasticity tensor $\mathbb{C}$: $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

Elastic properties of tissues

- $\mu = 0$: dominant wave type is a compressional wave.
- Pressure $p = \lambda \nabla \cdot u \quad$ in $\Omega \times \mathbb{R}_+$.
- Acoustic wave equation:

$$\begin{cases} \dfrac{1}{\lambda}\dfrac{\partial^2 p}{\partial t^2} - \nabla \cdot \dfrac{1}{\rho} \nabla p = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ p = F \cdot \nu & \text{on } \partial\Omega \times \mathbb{R}_+, \\ p(x,0) = \dfrac{\partial p}{\partial t}(x,0) = 0 & \text{in } \Omega. \end{cases}$$

- Time-harmonic regime:

$$\begin{cases} \nabla \cdot \dfrac{1}{\rho} \nabla \hat{p} + \dfrac{\omega^2}{\lambda} \hat{p} = 0 & \text{in } \Omega, \\ \hat{p} = \hat{F} \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

- Density ρ: ultrasound imaging.

Elastic properties of tissues

- Time harmonic regime:

$$\begin{cases} 2\nabla \cdot \mu \nabla^s u + \nabla \lambda \nabla \cdot u + \omega^2 \rho u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = F & \text{on } \partial\Omega. \end{cases}$$

- Shear modulus μ: stiffness depends on the tissue composition; related to abnormal phatological processes.
- Compressional modulus λ: 4 order of magnitude larger than μ.
- Modified Stokes system as $\lambda \to +\infty$:

$$\begin{cases} 2\nabla \cdot \mu \nabla^s u + \nabla p + \omega^2 \rho u = 0 & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ p\nu + \mu \dfrac{\partial u}{\partial \nu} = F & \text{on } \partial\Omega. \end{cases}$$

- Remove λ from consideration: reduce the Lamé system to the modified Stokes system.
- Viscosity properties: $\Re\mu$ and $\Im\mu$ related by Kramers-Kronig relations.