

Lecture 3: Layer potential techniques

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Layer potential techniques

- Layer potential techniques:
 - Conductivity equation;
 - Helmholtz equation.

Layer potential techniques

- **Fundamental solution to the Laplacian:**

$$\Gamma_0(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & d = 2, \\ \frac{1}{(2-d)\omega_d} |x|^{2-d}, & d \geq 3, \end{cases}$$

- ω_d : area of the unit sphere in \mathbb{R}^d .
- Ω : bounded domain in \mathbb{R}^d , $d \geq 2$, of class $\mathcal{C}^{1,\eta}$ for some $\eta > 0$.
- $\nu(y)$: outward unit normal to $\partial\Omega$ at y .

Layer potential techniques

- **Single-** and **double-layer potentials** of $\varphi \in L^2(\partial\Omega)$:

$$\mathcal{S}_\Omega^0[\varphi](x) := \int_{\partial\Omega} \Gamma_0(x-y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_\Omega^0[\varphi](x) := \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Gamma_0(x-y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial\Omega.$$

- **Neumann-Poincaré operator**: $\mathcal{K}_\Omega^0 : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$:

$$\mathcal{K}_\Omega^0[\varphi](x) := \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y-x, \nu(y) \rangle}{|x-y|^d} \varphi(y) d\sigma(y).$$

- $(\mathcal{K}_\Omega^0)^*$: L^2 -adjoint of \mathcal{K}_Ω^0 .
- \mathcal{K}_Ω^0 and $(\mathcal{K}_\Omega^0)^*$: **compact** in $L^2(\partial\Omega)$.

Layer potential techniques

- **Jump relations:** For $\varphi \in L^2(\partial\Omega)$,

$$(\mathcal{D}_\Omega^0[\varphi])|_{\pm}(x) = \left(\mp \frac{1}{2}I + \mathcal{K}_\Omega^0\right)[\varphi](x) \quad \text{a.e. } x \in \partial\Omega;$$

$$\mathcal{S}_\Omega^0[\varphi]\Big|_+(x) = \mathcal{S}_\Omega^0[\varphi]\Big|_-(x) \quad \text{a.e. } x \in \partial\Omega;$$

$$\frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_{\pm}(x) = \left(\pm \frac{1}{2}I + (\mathcal{K}_\Omega^0)^*\right)[\varphi](x) \quad \text{a.e. } x \in \partial\Omega.$$

- For $\varphi \in L^2(\partial\Omega)$, $\partial\mathcal{D}_\Omega^0[\varphi]/\partial\nu$ exists (in $H^{-1}(\partial\Omega)$) and has no jump across $\partial\Omega$:

$$\frac{\partial}{\partial\nu}\mathcal{D}_\Omega^0[\varphi]\Big|_+ = \frac{\partial}{\partial\nu}\mathcal{D}_\Omega^0[\varphi]\Big|_-.$$

- For $\varphi \in L^2(\partial\Omega)$,

$$\frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_+(x) - \frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_-(x) = \varphi(x) \quad \text{a.e. } x \in \partial\Omega.$$

Layer potential techniques

- **Dirichlet-to-Neumann operator** $\mathcal{N} : L^2(\partial\Omega) \rightarrow H^{-1}(\partial\Omega)$:

$$\mathcal{N}[\varphi] = \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega};$$

- u : solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

- Identity:

$$\frac{\partial}{\partial \nu} \mathcal{D}_{\Omega}^0[\varphi] \Big|_{\pm} = \left(\frac{1}{2} + (\mathcal{K}_{\Omega}^0)^* \right) \mathcal{N}[\varphi].$$

Layer potential techniques

- **Capacity:**

- $d = 2$; $(\varphi_e, a) \in L^2(\partial\Omega) \times \mathbb{R}$:

$$\begin{cases} \frac{1}{2\pi} \int_{\partial\Omega} \ln|x-y| \varphi_e(y) d\sigma(y) + a = 0 & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \varphi_e(y) d\sigma(y) = 1. \end{cases}$$

- **Logarithmic capacity** of $\partial\Omega$: $\text{cap}(\partial\Omega) := e^{2\pi a}$.
- $d = 3$; $\varphi_e \in L^2(\partial\Omega)$:

$$\begin{cases} \int_{\partial\Omega} \frac{\varphi_e(y)}{|x-y|} d\sigma(y) = \text{constant} & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \varphi_e(y) d\sigma(y) = 1. \end{cases}$$

- **Capacity** of $\partial\Omega$: $\frac{1}{\text{cap}(\partial\Omega)} := -S_{\Omega}^0[\varphi_e]$.

Layer potential techniques

- **Spectrum of the Neumann–Poincaré Operator:**

- $(\mathcal{K}_\Omega^0)^* : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$.
- Spectrum of $(\mathcal{K}_\Omega^0)^*$:

$$\sigma(\mathcal{K}_\Omega^0)^* \subset (-1/2, 1/2].$$

- $(1/2)I + \mathcal{K}_\Omega^0$: **invertible** on $L^2(\partial\Omega)$.
- $-(1/2)I + \mathcal{K}_\Omega^0$: **invertible** on $L_0^2(\partial\Omega)$.
- $L_0^2(\partial\Omega) := \left\{ \varphi \in L^2(\partial\Omega) : \int_{\partial\Omega} \varphi \, d\sigma = 0 \right\}$.

Layer potential techniques

- Proof by contradiction:
 - $\lambda \in (-\infty, -1/2] \cup (1/2, +\infty)$; $\varphi \in L^2(\partial\Omega)$ satisfies $(\lambda I - (\mathcal{K}_\Omega^0)^*)[\varphi] = 0$ and $\varphi \neq 0$.
 - $\mathcal{K}_\Omega^0[1] = 1/2 \Rightarrow$

$$0 = \int_{\partial\Omega} (\lambda I - (\mathcal{K}_\Omega^0)^*)[\varphi] d\sigma = \int_{\partial\Omega} \varphi(\lambda - \frac{1}{2}) d\sigma.$$

- $\Rightarrow \int_{\partial\Omega} \varphi d\sigma = 0$.
- $\Rightarrow \mathcal{S}_\Omega^0[\varphi](x) = O(|x|^{1-d})$ and $\nabla \mathcal{S}_\Omega^0[\varphi](x) = O(|x|^{-d})$, $|x| \rightarrow +\infty$ for $d \geq 2$.
- $\varphi \neq 0 \Rightarrow (A, B)$ cannot be zero:

$$A = \int_{\Omega} |\nabla \mathcal{S}_\Omega^0[\varphi]|^2 dx \text{ and } B = \int_{\mathbb{R}^d \setminus \bar{\Omega}} |\nabla \mathcal{S}_\Omega^0[\varphi]|^2 dx.$$

- By contradiction: if A and B are zero, then $\mathcal{S}_\Omega^0[\varphi] = \text{constant}$ in Ω and in $\mathbb{R}^d \setminus \bar{\Omega} \Rightarrow \varphi = 0$.

Layer potential techniques

- Divergence theorem \Rightarrow

$$A = \int_{\partial\Omega} \left(-\frac{1}{2}I + (\mathcal{K}_\Omega^0)^*\right)[\varphi] \mathcal{S}_\Omega^0[\varphi] d\sigma \text{ and } B = - \int_{\partial\Omega} \left(\frac{1}{2}I + (\mathcal{K}_\Omega^0)^*\right)[\varphi] \mathcal{S}_\Omega^0[\varphi] d\sigma.$$

- $(\lambda I - (\mathcal{K}_\Omega^0)^*)[\varphi] = 0 \Rightarrow$

$$\lambda = \frac{1}{2} \frac{B - A}{B + A} \Rightarrow |\lambda| < 1/2 \Rightarrow \text{contradiction.}$$

- For $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, $\lambda I - (\mathcal{K}_\Omega^0)^*$: **one to one on $L^2(\partial\Omega)$** .

- If $\lambda = 1/2$, then $A = 0 \Rightarrow \mathcal{S}_\Omega^0[\varphi] = \text{constant in } \Omega$.

- \Rightarrow

- $\mathcal{S}_\Omega^0[\varphi]$: harmonic in $\mathbb{R}^d \setminus \partial\Omega$;
- $\mathcal{S}_\Omega^0[\varphi](x) = O(|x|^{1-d})$, $|x| \rightarrow +\infty$ (since $\varphi \in L_0^2(\partial\Omega)$);
- $\mathcal{S}_\Omega^0[\varphi]$: constant on $\partial\Omega$.

- $(\mathcal{K}_\Omega^0)^*[\varphi] = (1/2)\varphi \Rightarrow$

$$B = - \int_{\partial\Omega} \varphi \mathcal{S}_\Omega^0[\varphi] d\sigma = C \int_{\partial\Omega} \varphi d\sigma = 0,$$

- $\Rightarrow \varphi = 0 \Rightarrow (1/2)I - (\mathcal{K}_\Omega^0)^*$: **one to one on $L_0^2(\partial\Omega)$** .

Layer potential techniques

- **Symmetrization of $(\mathcal{K}_\Omega^0)^*$:**
 - **Non-self-adjoint** operator $(\mathcal{K}_\Omega^0)^*$: can be realized as a **self-adjoint** operator on $H^{-1/2}(\partial\Omega)$ by introducing a **new inner product**.
 - \mathcal{S}_Ω^0 in $H^{-1/2}(\partial\Omega)$: **self-adjoint** and $-\mathcal{S}_\Omega^0 \geq 0$ on $H^{-1/2}(\partial\Omega)$.
 - $(\mathcal{K}_\Omega^0)^* : H^{-1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$: **compact**.
 - **Calderón identity** \Rightarrow

$$\mathcal{S}_\Omega^0(\mathcal{K}_\Omega^0)^* = \mathcal{K}_\Omega^0\mathcal{S}_\Omega^0 \quad \text{on } H^{-1/2}(\partial\Omega).$$

Layer potential techniques

- **Kernel** of \mathcal{S}_Ω^0 :
 - $d \geq 3$; $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ has a **bounded inverse**.
 - $d = 2$; If $\phi_0 \in \text{Ker}(\mathcal{S}_\Omega^0)$, then u :

$$u(x) := \mathcal{S}_\Omega^0[\phi_0](x), \quad x \in \mathbb{R}^2$$

satisfies $u = 0$ on $\partial\Omega \Rightarrow u(x) = 0$ for all $x \in \Omega$.

- **Jump condition** \Rightarrow

$$(\mathcal{K}_\Omega^0)^*[\phi_0] = \frac{1}{2}\phi_0 \quad \text{on } \partial\Omega.$$

- If $\langle \chi(\partial\Omega), \phi_0 \rangle_{1/2, -1/2} = 0$, then $u(x) \rightarrow 0$ as $|x| \rightarrow \infty \Rightarrow u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega \Rightarrow \phi_0 = 0$.
- Eigenfunctions: **one dimensional subspace** of $H^{-1/2}(\partial\Omega)$.
- $\Rightarrow \text{Ker}(\mathcal{S}_\Omega^0)$: of **at most one dimension**.
- $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ has a **bounded inverse** iff **$\text{In cap}(\partial\Omega) \neq 0$** .

Layer potential techniques

- $d = 3$; **inner product**:

$$\langle u, v \rangle_{\mathcal{H}^*} = -\langle \mathcal{S}_{\Omega}^0[v], u \rangle_{\frac{1}{2}, -\frac{1}{2}},$$

- **Equivalent**: $H^{-1/2}(\partial\Omega)$.
- $(\mathcal{K}_{\Omega}^0)^*$: **self-adjoint** in $\mathcal{H}^*(\partial\Omega)$;
- (λ_j, φ_j) , $j = 0, 1, 2, \dots$: eigenvalue and normalized eigenfunction pair of $(\mathcal{K}_{\Omega}^0)^*$ in $\mathcal{H}^*(\partial\Omega)$ with $\lambda_0 = 1/2 \leftarrow \mathcal{K}_{\Omega}^0[1] = 1/2$.
- $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ for $j \geq 1$ with $|\lambda_1| \geq |\lambda_2| \geq \dots \rightarrow 0$ as $j \rightarrow \infty$;
- **Spectral representation formula**: for any $\psi \in H^{-1/2}(\partial\Omega)$,

$$(\mathcal{K}_{\Omega}^0)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j \langle \varphi_j, \psi \rangle_{\mathcal{H}^*} \varphi_j.$$

- $\mathcal{H}(\partial\Omega)$: $H^{1/2}(\partial\Omega)$ equipped with the equivalent inner product

$$\langle u, v \rangle_{\mathcal{H}} = \langle v, (-\mathcal{S}_{\Omega}^0)^{-1}[u] \rangle_{\frac{1}{2}, -\frac{1}{2}}.$$

- \mathcal{S}_{Ω}^0 : **isometry** between $\mathcal{H}^*(\partial\Omega)$ and $\mathcal{H}(\partial\Omega)$.

Layer potential techniques

- $d = 2$; $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$: **not injective** (in general).
- **Substitute:**

$$\tilde{\mathcal{S}}_\Omega[\psi] := \begin{cases} \mathcal{S}_\Omega^0[\psi] & \text{if } \langle \chi(\partial\Omega), \psi \rangle_{\frac{1}{2}, -\frac{1}{2}} = 0, \\ -\chi(\partial\Omega) & \text{if } \psi = \varphi_0, \end{cases}$$

- φ_0 : unique eigenfunction of $(\mathcal{K}_\Omega^0)^*$ associated with eigenvalue $1/2$ s.t. $\langle \chi(\partial\Omega), \varphi_0 \rangle_{\frac{1}{2}, -\frac{1}{2}} = 1$.
- $\tilde{\mathcal{S}}_\Omega : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$: **invertible**.
- **Calderón identity:**

$$\mathcal{K}_\Omega^0 \tilde{\mathcal{S}}_\Omega = \tilde{\mathcal{S}}_\Omega (\mathcal{K}_\Omega^0)^*.$$

Layer potential techniques

- $(\mathcal{K}_\Omega^0)^*$: **compact self-adjoint** in $\mathcal{H}^*(\partial\Omega)$ equipped with

$$\langle u, v \rangle_{\mathcal{H}^*} = -\langle \tilde{\mathcal{S}}_\Omega[v], u \rangle_{\frac{1}{2}, -\frac{1}{2}}.$$

- (λ_j, φ_j) , $j = 0, 1, 2, \dots$: eigenvalue and normalized eigenfunction pair of $(\mathcal{K}_\Omega^0)^*$ with $\lambda_0 = \frac{1}{2}$. $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ with $|\lambda_1| \geq |\lambda_2| \geq \dots \rightarrow 0$ as $j \rightarrow \infty$;
- **Twin property**: For any $j \geq 1$, $\pm\lambda_j$: eigenvalues of $(\mathcal{K}_\Omega^0)^*$;
- $\mathcal{H}^*(\partial\Omega) = \mathcal{H}_0^*(\partial\Omega) \oplus \{\mu\varphi_0, \mu \in \mathbb{C}\}$, where $\mathcal{H}_0^*(\partial\Omega)$: zero mean subspace of $\mathcal{H}^*(\partial\Omega)$;
- For any $\psi \in H^{-1/2}(\partial\Omega)$,

$$(\mathcal{K}_\Omega^0)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j \langle \varphi_j, \psi \rangle_{\mathcal{H}^*} \varphi_j.$$

- $\mathcal{H}(\partial\Omega)$: $H^{1/2}(\partial\Omega)$ equipped with the equivalent inner product:

$$\langle u, v \rangle_{\mathcal{H}} = \langle v, -\tilde{\mathcal{S}}_\Omega^{-1}[u] \rangle_{\frac{1}{2}, -\frac{1}{2}}.$$

- $\tilde{\mathcal{S}}_\Omega$: **isometry** between $\mathcal{H}^*(\partial\Omega)$ and $\mathcal{H}(\partial\Omega)$.

Layer potential techniques

- **Neumann function**: For Ω : smooth bounded domain in \mathbb{R}^d ,

$$\left\{ \begin{array}{l} -\Delta_x N(x, z) = \delta_z \quad \text{in } \Omega, \\ \frac{\partial N}{\partial \nu_x} \Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|}, \int_{\partial\Omega} N(x, z) d\sigma(x) = 0 \quad \text{for } z \in \Omega. \end{array} \right.$$

- $U(x) := \int_{\partial\Omega} N(x, z)g(z) d\sigma(z)$: solution to

$$\left\{ \begin{array}{l} \Delta U = 0 \quad \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} \Big|_{\partial\Omega} = g, \\ \int_{\partial\Omega} U d\sigma = 0. \end{array} \right.$$

Layer potential techniques

- N : symmetric in its arguments, $N(x, z) = N(z, x)$ for $x \neq z \in \Omega$.

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$$N(x, z) = \begin{cases} -\frac{1}{2\pi} \ln |x - z| + R_2(x, z) & \text{if } d = 2, \\ \frac{1}{(d-2)\omega_d} \frac{1}{|x - z|^{d-2}} + R_d(x, z) & \text{if } d \geq 3, \end{cases}$$

- $R_d(\cdot, z) \in H^{\frac{3}{2}}(\Omega)$ for any $z \in \Omega$, $d \geq 2$.

Layer potential techniques

- For $D \Subset \Omega$,

$$N_D[f](x) := \int_{\partial D} N(x, y) f(y) d\sigma(y), \quad x \in \Omega.$$

- Relation between the fundamental solution Γ_0 and the Neumann function N :

For $z \in \Omega$ and $x \in \partial\Omega$, let $\Gamma_z(x) := \Gamma_0(x - z)$ and $N_z(x) := N(x, z)$.

Then

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega^0\right)[N_z](x) = \Gamma_z(x) \quad \text{modulo constants, } x \in \partial\Omega.$$

Layer potential techniques

- **Outgoing fundamental solution** $\Gamma_\omega(x)$ to the Helmholtz operator $\Delta + \omega^2$ in \mathbb{R}^d , $d = 2, 3$: $(\Delta_x + \omega^2)\Gamma_\omega(x) = \delta_0(x)$

$$\Gamma_\omega(x) = \begin{cases} -\frac{i}{4} H_0^{(1)}(\omega|x|), & d = 2, \\ -\frac{e^{i\omega|x|}}{4\pi|x|}, & d = 3, \end{cases}$$

- $H_0^{(1)}$: Hankel function of the first kind of order 0.
- Behavior of $H_0^{(1)}$ near 0:

$$-\frac{i}{4} H_0^{(1)}(\omega|x|) \sim \frac{1}{2\pi} \ln|x| \quad \left[\Delta_x \left(\frac{1}{2\pi} \ln|x| \right) = \delta_0(x) \right].$$

- **Sommerfeld radiation condition:**

$$\left| \frac{x}{|x|} \cdot \nabla \Gamma_\omega(x) - i\omega \Gamma_\omega(x) \right| = \begin{cases} O(|x|^{-3/2}), & d = 2, \\ O(|x|^{-2}), & d = 3. \end{cases}$$

Layer potential techniques

- **Single- and double-layer potentials:** For $\varphi \in L^2(\partial D)$,

$$\mathcal{S}_D^\omega[\varphi](x) = \int_{\partial D} \Gamma_\omega(x-y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_D^\omega[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(y)} \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D.$$

- $\Gamma_\omega(x)$ **outgoing fundamental solution to the Helmholtz operator** \Rightarrow
 - $\mathcal{S}_D^\omega[\varphi]$ and $\mathcal{D}_D^\omega[\varphi]$ satisfy the Helmholtz equation

$$(\Delta + \omega^2)u = 0 \quad \text{in } D \text{ and in } \mathbb{R}^d \setminus \bar{D}.$$

- $\mathcal{S}_D^\omega[\varphi]$ and $\mathcal{D}_D^\omega[\varphi]$ satisfy the Sommerfeld radiation condition.

Layer potential techniques

- **Jump relations:** For $\varphi \in L^2(\partial D)$,

$$\left. \frac{\partial(\mathcal{S}_D^\omega[\varphi])}{\partial\nu} \right|_{\pm}(x) = \left(\pm \frac{1}{2}I + (\mathcal{K}_D^\omega)^* \right) [\varphi](x) \quad \text{a.e. } x \in \partial D,$$
$$\left. (\mathcal{D}_D^\omega[\varphi]) \right|_{\pm}(x) = \left(\mp \frac{1}{2}I + \mathcal{K}_D^\omega \right) [\varphi](x) \quad \text{a.e. } x \in \partial D,$$

- \mathcal{K}_D^ω and $(\mathcal{K}_D^\omega)^*$:

$$\mathcal{K}_D^\omega[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial\nu(y)} \varphi(y) d\sigma(y);$$
$$(\mathcal{K}_D^\omega)^*[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial\nu(x)} \varphi(y) d\sigma(y).$$

- $(\mathcal{K}_D^\omega)^*$: L^2 -adjoint of $\mathcal{K}_D^{-\omega}$ (complex inner product).
- \mathcal{K}_D^ω and $(\mathcal{K}_D^\omega)^*$: **compact** on $L^2(\partial D)$.

Layer potential techniques

- Ω : smooth bounded domain in \mathbb{R}^d ; ω^2 : **not a Dirichlet eigenvalue** in Ω .
- **Dirichlet function:**

$$\begin{cases} \Delta_x G_\omega(x, z) + \omega^2 G_\omega(x, z) = \delta_z & \text{in } \Omega, \\ G_\omega(x, z)|_{\partial\Omega} = 0 & \text{for } z \in \Omega. \end{cases}$$

- $V(x) := \int_{\partial\Omega} G_\omega(x, z) f(z) d\sigma(z)$: solution to

$$\begin{cases} \Delta V + \omega^2 V = 0 & \text{in } \Omega, \\ V|_{\partial\Omega} = f. \end{cases}$$

- Relation between Γ_ω and G_ω :

$$\left(\frac{1}{2}I + (\mathcal{K}_\Omega^\omega)^* \right) \left[\frac{\partial G_\omega}{\partial \nu} \right] = \frac{\partial \Gamma_\omega}{\partial \nu}.$$