Lecture 4: Scale separation techniques

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Scale-Separation techniques

- Scale separation techniques: take advantage of the smallness of the imaged anomalies.
- Small-volume expansions of measured data:
  - Derivations based on layer potential techniques.
  - Extraction of geometric and material features of the anomaly from measured data.
  - Notions of generalized polarization tensors and scattering coefficients: building blocks of the small-volume expansions.
- Direct imaging algorithms: subspace projection algorithms.

Scale-Separation techniques

- Conductivity problem in free space
  - B: bounded smooth domain in $\mathbb{R}^d$; $O \in B$.
  - $0 < k \neq 1 < +\infty$ and $\lambda(k) := (k+1)/(2(k-1))$.
  - h: harmonic function in $\mathbb{R}^d$; u:

$$\begin{cases} \nabla \cdot ((1 + (k-1)\chi(B))\nabla u_k) = 0 \quad \text{in } \mathbb{R}^d, \\ u_k(x) - h(x) = O(|x|^{1-d}) \quad \text{as } |x| \to +\infty. \end{cases}$$

  - Integral representation formula:

$$u_k(x) = h(x) + \mathcal{S}_B^0(\lambda(k)I - (\mathcal{K}_B^0)^*)^{-1}[\frac{\partial h}{\partial \nu}|_{\partial B}](x) \quad \text{for } x \in \mathbb{R}^d.$$

  - $\lambda(k) := (k+1)/(2(k-1))$: conductivity contrast.

Scale-Separation techniques

- Integral representation formula:

$$u_k(x) = h(x) + \mathcal{S}_B^0[\phi](x).$$

- Transmission conditions:
  - $u_k|_+ = u_k|_-$ on ∂B.
  - $k\frac{\partial u_k}{\partial \nu}|_+ = \frac{\partial u_k}{\partial \nu}|_-$ on ∂B.
- $\Rightarrow$

$$\phi = (\lambda(k)I - (\mathcal{K}_B^0)^*)^{-1}[\frac{\partial h}{\partial \nu}|_{\partial B}].$$

Scale-Separation techniques

- Taylor's formula expansion:

$$\Gamma_0(x-y) = \sum_{\alpha, |\alpha|=0}^{+\infty} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha \Gamma_0(x) y^\alpha, \quad y \text{ in a compact set}, \quad |x| \to +\infty.$$

- Far-field expansion of $(u_k - h)(x)$ as $|x| \to +\infty$:

$$\sum_{|\alpha|,|\beta|=1}^{+\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial_x^\alpha \Gamma_0(x) \partial^\beta h(0) \int_{\partial B} (\lambda(k) I - (\mathcal{K}_B^0)^*)^{-1} \left[\nu(x) \cdot \nabla x^\alpha\right](y) y^\beta \, d\sigma(y).$$

- For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$: $\partial^\alpha f = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} f$ and $x^\alpha := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$.
- h: entire harmonic function in $\mathbb{R}^d$.

Scale-Separation techniques

- Generalized polarization tensor $M_{\alpha\beta}$, $\alpha, \beta \in \mathbb{N}^d$:

$$M_{\alpha\beta}(\lambda(k), B) := \int_{\partial B} y^\beta \phi_\alpha(y)\, d\sigma(y);$$

- ϕ_α:

$$\phi_\alpha(y) := \left(\lambda(k)I - (\mathcal{K}_B^0)^*\right)^{-1}\left[\nu(x) \cdot \nabla x^\alpha\right](y), \quad y \in \partial B.$$

- For $|\alpha| = |\beta| = 1$, $M = (m_{pq})_{p,q=1}^d$ polarization tensor

$$m_{pq} := \int_{\partial B} y_q \left(\lambda(k)I - (\mathcal{K}_B^0)^*\right)^{-1}[\nu_p](y)\, d\sigma(y),$$

- $\nu = (\nu_1, \ldots, \nu_d)$.
- Generalized polarization tensors $\Rightarrow$ complete information about the far-field expansion of u.

Scale-Separation techniques

- Spectral representation formula for $(\mathcal{K}_B^0)^* \Rightarrow$

$$(\lambda(k)I - (\mathcal{K}_B^0)^*)^{-1}[\psi] = \sum_{j=0}^{\infty} \frac{\langle \psi, \varphi_j \rangle_{\mathcal{H}^*} \varphi_j}{\lambda(k) - \lambda_j},$$

- (λ_j, φ_j): eigenvalues and eigenvectors of $(\mathcal{K}_B^0)^*$ in $\mathcal{H}^*$.
- Decomposition of the entries of the polarization tensor:

$$m_{pq}(\lambda(k), B) = \sum_{j=1}^{\infty} \frac{\langle \nu_p, \varphi_j \rangle_{\mathcal{H}^*} \langle \varphi_j, x_q \rangle_{-\frac{1}{2},\frac{1}{2}}}{\lambda(k) - \lambda_j}.$$

- $\langle \nu_p, \chi(\partial B) \rangle_{-\frac{1}{2},\frac{1}{2}} = 0$. $\lambda_0 = 1/2 \Rightarrow \langle \nu_p, \varphi_0 \rangle_{\mathcal{H}^*} = 0$.

Scale-Separation techniques

- From:

$$\langle \varphi_j, x_q \rangle_{-\frac{1}{2},\frac{1}{2}} = \Big\langle \big(\tfrac{1}{2} - \lambda_j\big)^{-1} \big(\tfrac{1}{2} I - (\mathcal{K}_B^0)^*\big)[\varphi_j], x_q \Big\rangle_{-\frac{1}{2},\frac{1}{2}}$$

$$= \frac{-1}{1/2 - \lambda_j} \Big\langle \frac{\partial \mathcal{S}_B^0[\varphi_j]}{\partial \nu}\Big|_-, x_q \Big\rangle_{-\frac{1}{2},\frac{1}{2}}$$

$$= \frac{-1}{1/2 - \lambda_j} \Big[\int_{\partial B} \frac{\partial x_q}{\partial \nu} \mathcal{S}_B^0[\varphi_j] d\sigma - \int_B \Big(\Delta x_q \mathcal{S}_B^0[\varphi_j] - x_q \Delta \mathcal{S}_B^0[\varphi_j] \Big) dx \Big]$$

$$= \frac{\langle \nu_q, \varphi_j \rangle_{\mathcal{H}^*}}{1/2 - \lambda_j} .$$

- $\Rightarrow$

$$m_{pq}(\lambda(k), B) = \sum_{j=1}^{\infty} \frac{\langle \nu_p, \varphi_j \rangle_{\mathcal{H}^*} \langle \nu_q, \varphi_j \rangle_{\mathcal{H}^*}}{(1/2 - \lambda_j)(\lambda(k) - \lambda_j)} .$$

Scale-Separation techniques

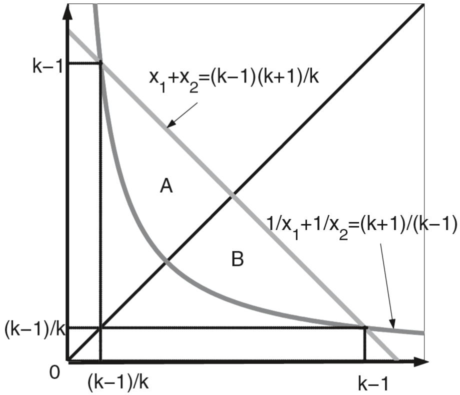
- Properties of the polarization tensor:
  - $M(\lambda(k), B)$: symmetric;
  - $M(\lambda(k), B)$: positive definite if $k > 1$;
  - $M(\lambda(k), B)$: negative definite if $0 < k < 1$.
  - Optimal bounds:

$$\frac{1}{k-1} \operatorname{tr}(M(\lambda(k), B)) < (1 + \frac{1}{k})|B|$$

and

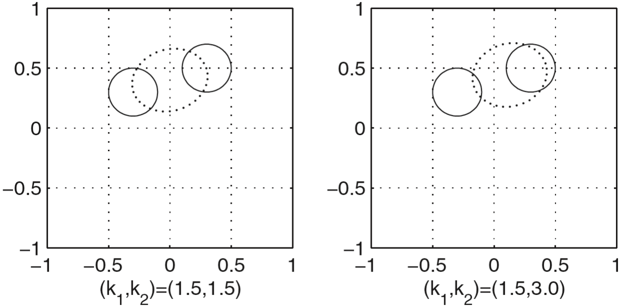
$$(k-1) \operatorname{tr}(M(\lambda(k), B)^{-1}) \leq \frac{(1+k)}{|B|}.$$

Scale-Separation techniques



Optimal bounds for the polarization tensor.

Scale-Separation techniques

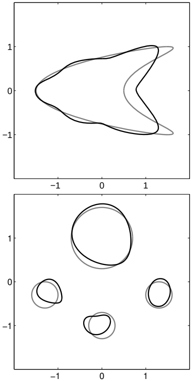


Equivalent polarization tensor.

Scale-Separation techniques

Properties of high-order polarization tensors:

- Recover high-frequency information on the shape;
- Separate topology;
- Determine uniquely the shape and the material parameter.



Scale-Separation techniques

- Positivity and symmetry properties on harmonic coefficients; optimal bounds.
- Harmonic coefficients:

$$(x_1 + ix_2)^m = \sum_{|\alpha|=m} a_\alpha^m x^\alpha + i \sum_{|\beta|=m} b_\beta^m x^\beta .$$

- I, J: finite index sets, $\{a_\alpha\}, \{\beta\}$: harmonic coefficients,

$$\sum_{\alpha\in I, \beta\in J} M_{\alpha\beta} a_\alpha b_\beta = \sum_{\alpha\in I, \beta\in J} M_{\alpha\beta} a_\alpha b_\beta , \quad \sum_{\alpha,\beta\in I} M_{\beta\alpha} a_\alpha a_\beta > 0 \ (k > 1).$$

- Translation, rotation, and scaling formulas.

Scale-Separation techniques

- Conductivity anomaly D inside a background medium Ω.
- k: conductivity of D; $0 < k \neq 1 < +\infty$; 1: background conductivity.
- $\lambda = (k+1)/(2(k-1))$: conductivity contrast.
- Detect, localize, and characterize the anomaly D from boundary measurements on $\partial\Omega$.

Scale-Separation techniques

- Representation formula
- For $g \in L_0^2(\partial\Omega)$, electric potential u:

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi(D)\Big)\nabla u = 0 \quad \text{in } \Omega \,, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g \,, \\ \displaystyle\int_{\partial\Omega} u(x)\, d\sigma(x) = 0 \,, \end{cases}$$

- Background solution U:

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega \,, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g \,, \\ \displaystyle\int_{\partial\Omega} U(x)\, d\sigma(x) = 0 \,. \end{cases}$$

Scale-Separation techniques

- Decomposition formula of u into a harmonic part and a refraction part:

$$u(x) = H(x) + \mathcal{S}_D^0[\phi](x), \quad x \in \mathbb{R}^d \setminus \partial\Omega \, ;$$

- Harmonic function H:

$$H(x) = -\mathcal{S}_\Omega^0[g](x) + \mathcal{D}_\Omega^0[f](x), \quad x \in \mathbb{R}^d \setminus \partial\Omega \, , \quad f := u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \, ;$$

- $\phi \in L_0^2(\partial D)$:

$$\left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_D^0)^* \right)[\phi] = \frac{\partial H}{\partial \nu}\bigg|_{\partial D} \quad \text{on } \partial D \, .$$

Scale-Separation techniques

- The decomposition into a harmonic part and a refraction part: unique.
- $\forall\, n \in \mathbb{N}$, there exists a constant $C_n = C(n, \Omega, \mathrm{dist}(D, \partial\Omega))$ independent of D and the conductivity k s.t.

$$\|H\|_{\mathcal{C}^n(\overline{D})} \leq C_n \|g\|_{L^2(\partial\Omega)} \,.$$

- $$H(x) + \mathcal{S}_D[\phi](x) = 0, \quad \forall\, x \in \mathbb{R}^d \setminus \overline{\Omega} \,.$$

Scale-Separation techniques

- Harmonic part H:

$$H(x) = \begin{cases} u(x) - (k-1) \int_D \nabla_y \Gamma_0(x-y) \cdot \nabla u(y)\, dy, & x \in \Omega\,, \\ -(k-1) \int_D \nabla_y \Gamma_0(x-y) \cdot \nabla u(y)\, dy, & x \in \mathbb{R}^d \setminus \overline{\Omega}\,. \end{cases}$$

- Proof:
  - Jump formula $\Rightarrow$ on ∂D:

$$\frac{\partial u}{\partial \nu}\Big|_- = \frac{\partial H}{\partial \nu} + \frac{\partial}{\partial \nu} \mathcal{S}_D^0[\phi]\Big|_- = \frac{\partial H}{\partial \nu} + (-\frac{1}{2} I + (\mathcal{K}_D^0)^*)[\phi] = \frac{1}{k-1}\phi\,.$$

  - Green's formula.

Scale-Separation techniques

- For $g \in L^2_0(\partial\Omega)$,

$$U(y) := \int_{\partial\Omega} N(x,y) g(x)\, d\sigma(x)\,.$$

- Representation formula:

$$u(x) = U(x) - N_D[\phi](x), \quad x \in \partial\Omega.$$

- Proof:
  - Representation formula $\Rightarrow x \in \Omega$:

$$H(x) = -\mathcal{S}^0_\Omega[g](x) + \mathcal{D}^0_\Omega\Big[H|_{\partial\Omega} + \mathcal{S}^0_D[\phi]|_{\partial\Omega}\Big](x)\,.$$

  - Jump formula $\Rightarrow$ on $\partial\Omega$:

$$\left(\frac{1}{2}I - \mathcal{K}^0_\Omega\right)[H|_{\partial\Omega}] = -\mathcal{S}^0_\Omega[g]|_{\partial\Omega} + \left(\frac{1}{2}I + \mathcal{K}^0_\Omega\right)[\mathcal{S}^0_D[\phi]|_{\partial\Omega}]\,.$$

  - $U = -\mathcal{S}^0_\Omega[g] + \mathcal{D}^0_\Omega[U|_{\partial\Omega}]$ in $\Omega \Rightarrow$ (by Green's formula)

$$\left(\frac{1}{2}I - \mathcal{K}^0_\Omega\right)[U|_{\partial\Omega}] = -\mathcal{S}^0_\Omega[g]|_{\partial\Omega}\,.$$

Scale-Separation techniques

- $\phi \in L^2_0(\partial D) \Rightarrow$
$$-\left(\frac{1}{2}I - \mathcal{K}^0_\Omega\right)[N_D[\phi]]|_{\partial\Omega}] = (\mathcal{S}^0_D[\phi])|_{\partial\Omega} \; .$$
- $\Rightarrow$
$$\left(\frac{1}{2}I - \mathcal{K}^0_\Omega\right)\left[H|_{\partial\Omega} - U|_{\partial\Omega} + \left(\frac{1}{2}I + \mathcal{K}^0_\Omega\right)[N_D[\phi]|_{\partial\Omega}]\right] = 0 \; .$$
- $$H|_{\partial\Omega} - U|_{\partial\Omega} + \left(\frac{1}{2}I + \mathcal{K}^0_\Omega\right)[N_D[\phi]|_{\partial\Omega}] = C \text{ (constant).}$$
- $$(\frac{1}{2}I + \mathcal{K}^0_\Omega)[N_D[\phi]|_{\partial\Omega}] = (N_D[\phi])|_{\partial\Omega} + (\mathcal{S}^0_D[\phi])|_{\partial\Omega} \; .$$
- $$u|_{\partial\Omega} = U|_{\partial\Omega} - (N_D[\phi])|_{\partial\Omega} + C \; .$$
- All the functions $\in L^2_0(\partial\Omega) \Rightarrow C = 0$.

Scale-Separation techniques

- Energy estimates:

$$\|u - U\|_{L^2(\partial\Omega)} \leq \begin{cases} C(k-1)\,\|g\|_{L^2(\partial\Omega)}|D| & \text{if } k > 1\,, \\ C(\dfrac{1}{k} - 1)\,\|g\|_{L^2(\partial\Omega)}|D| & \text{if } 0 < k < 1\,. \end{cases}$$

- $H - U = \mathcal{D}_\Omega^0[u - U]$ in $\Omega \Rightarrow$

$$\|\nabla H - \nabla U\|_{L^\infty(\overline{D})} \leq C\|u - U\|_{L^2(\partial\Omega)} \leq C|k-1|\|g\|_{L^2(\partial\Omega)}|D|\,,$$

- C depends on $\mathrm{dist}(\partial\Omega, D)$.

Scale-Separation techniques

- Small-volume pointwise asymptotic expansion on $\partial\Omega$:

$$u(x) = U(x) - \delta^d \nabla U(z) \cdot M(\lambda, B) \nabla_z N(x, z) + O(\delta^{d+1}) \; ;$$

  - $D = \delta B + z$;
  - $O(\delta^{d+1})$: dominated by $C\delta^{d+1} \|g\|_{L^2(\partial\Omega)}$ for some C independent of $x \in \partial\Omega$;
  - U: background solution;
  - $N(x, z)$: Neumann function;
  - $M(\lambda, B) = (m_{pq})_{p,q=1}^d$: polarization tensor.
- Dipole approximation of the anomaly D.
- Generalized polarization tensors: building blocks of higher-order asymptotic expansions.

Scale-Separation techniques

- $\mu_0 > 0$ and $\varepsilon_0 > 0$: permeability and permittivity of the background medium Ω;
- $\mu_\star$ and $\varepsilon_\star$: permeability and the permittivity of D;
- $$\mu_\delta(x) = \begin{cases} \mu_0 \,, & x \in \Omega \setminus \overline{D} \,, \\ \mu_\star \,, & x \in D \,. \end{cases}$$
- Piecewise constant electric permittivity $\varepsilon_\delta(x)$: defined analogously.
- $\omega > 0$: given frequency.
- $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$; $K_\star = \omega\sqrt{\varepsilon_\star\mu_\star}$.
- $D = \delta B + z$.

Scale-Separation techniques

- u: the solution to the Helmholtz equation

$$\nabla \cdot (\frac{1}{\mu_\delta} \nabla u) + \omega^2 \varepsilon_\delta u = 0 \quad \text{in } \Omega \,,$$

with the boundary condition $u = f$ on $\partial\Omega$ with $f \in H^{\frac{1}{2}}(\partial\Omega)$.

- Equivalent formulation:

$$\begin{cases} (\Delta + \omega^2 \varepsilon_0 \mu_0) u = 0 \quad \text{in } \Omega \setminus \overline{D} \,, \\ (\Delta + \omega^2 \varepsilon_\star \mu_\star) u = 0 \quad \text{in } D \,, \\ \frac{1}{\mu_\star} \frac{\partial u}{\partial \nu}\Big|_{-} - \frac{1}{\mu_0} \frac{\partial u}{\partial \nu}\Big|_{+} = 0 \quad \text{on } \partial D \,, \\ u\big|_{-} - u\big|_{+} = 0 \quad \text{on } \partial D \,, \\ u = f \quad \text{on } \partial\Omega \,. \end{cases}$$

Scale-Separation techniques

- Assumption: $\omega^2 \varepsilon_0 \mu_0$ is not an eigenvalue for the operator $-\Delta$ in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions.
- $\Rightarrow$ existence and uniqueness of a solution u at least for δ small enough.
- U: background solution

$$\begin{cases} \Delta U + k_0^2 U = 0 & \text{in } \Omega\,, \\ U = f & \text{on } \partial\Omega\,. \end{cases}$$

Scale-Separation techniques

- Small-volume expansion: For any $x \in \partial\Omega$,

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial U}{\partial \nu}(x) + \delta^d \Bigg(\nabla U(z) \cdot M(\lambda, B) \frac{\partial \nabla_z G_{k_0}(x,z)}{\partial \nu_x} + k_0^2 (\frac{\varepsilon_\star}{\varepsilon_0} - 1) |B| U(z) \frac{\partial G_{k_0}(x,z)}{\partial \nu_x} \Bigg) + O(\delta^{d+1}) \; ;$$

- $M(\lambda, B)$: polarization tensor with

$$\lambda := \frac{(\mu_0/\mu_\star) + 1}{2((\mu_0/\mu_\star) - 1)} \; .$$

- G_{k_0}: the Dirichlet Green function.

Scale-Separation techniques

- Internal perturbations of $u_\cdot$: Small-volume expansion does not hold uniformly in Ω.
- Inner expansion:

$$u(x) \approx U(z) + \delta v(\frac{x-z}{\delta}, \frac{\mu_0}{\mu_\star}) \cdot \nabla U(z) \quad \text{for } x \text{ near } z \, ;$$

- $k = \mu_0/\mu_\star$; v:

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B} \, , \qquad \Delta v = 0 \quad \text{in } B \, , \\ v|_- - v|_+ = 0 \quad \text{on } \partial B \, , \\ k \dfrac{\partial v}{\partial \nu}\Big|_- - \dfrac{\partial v}{\partial \nu}\Big|_+ = 0 \quad \text{on } \partial B \, , \\ v(\xi) - \xi \to 0 \quad \text{as } |\xi| \to +\infty \, . \end{cases}$$

- Relation between v and M:

$$M(\lambda, B) = (k-1) \int_B \nabla v(\xi, k) \, d\xi.$$

Scale-Separation techniques

- Anomaly imaging algorithms
  - Direct imaging for the conductivity problem:
    - Projection-type algorithm;
    - MUSIC-type algorithm.
  - Direct imaging for the Helmholtz problem:
    - MUSIC-type algorithm;
    - Backpropagation-type algorithms.

Scale-Separation techniques

- Projection-type algorithm: Detection of a single anomaly.
- Apply a special type of current that makes ∇U constant in the inclusion D.
- Injection current $g = a \cdot \nu$ for a fixed unit vector $a \in \mathbb{R}^d \Rightarrow \nabla U = a$ in Ω.
- w: harmonic in $\Omega \Rightarrow$ Weighted boundary measurements $I_w[U]$:

$$I_w[U] := \int_{\partial\Omega} (u - U)(x) \frac{\partial w}{\partial \nu}(x)\, d\sigma(x) \approx -\delta^d \nabla U(z) \cdot M(\lambda, B) \nabla w(z)\,.$$

- $d = 2$ and D is a disk. Set

$$w(x) = -(1/2\pi) \log |x - y| \quad \text{for } y \in \mathbb{R}^2 \setminus \overline{\Omega}, x \in \Omega\,.$$

- $\Rightarrow$

$$I_w[U] \approx \frac{(k-1)|D|}{\pi(k+1)} \frac{(y - z) \cdot a}{|y - z|^2}\,, \quad y \in \mathbb{R}^2 \setminus \overline{\Omega}\,.$$

Scale-Separation techniques

- Location search algorithm:
    - Take two observation lines Σ_1 and Σ_2 contained in $\mathbb{R}^2 \setminus \overline{\Omega}$ given by
    $$\Sigma_1 := \text{ a line parallel to } a \,,$$
    $$\Sigma_2 := \text{ a line normal to } a \,.$$
    - Find two points $z_i^S \in \Sigma_i, i = 1, 2,$ s.t.
    $$I_w[U](z_1^S) = 0, \quad I_w[U](z_2^S) = \max_{y \in \Sigma_2} |I_w[U](y)| \,.$$
    - z^S: intersecting point of the two lines
    $$\Pi_1(z_1^S) := \{y \mid a \cdot (y - z_1^S) = 0\},$$
    $$\Pi_2(z_2^S) := \{y \mid (y - z_2^S) \text{ is parallel to } a\}.$$
    - z^S close to the center z of D: $|z^S - z| = O(\delta^2)$.
    - Once one locates the anomaly, $|D|(k-1)/(k+1)$: can be estimated.

Scale-Separation techniques

- A MUSIC-type algorithm: Detection of multiple anomalies
- P well-separated anomalies $D_p = \delta B_p + z_p$, with conductivities k_p, $p = 1, \ldots, P$.
- B_p: disks.
- $y_l \in \mathbb{R}^2 \setminus \Omega$ for $l = 1, \ldots, n$: source points.
- Set

$$U_{y_l} = w_{y_l} := -(1/2\pi) \log |x - y_l| \quad \text{for } x \in \Omega, \quad l = 1, \ldots, n \,.$$

Scale-Separation techniques

- MUSIC-type location search algorithm:
  - For $n \in \mathbb{N}$ sufficiently large, define the response matrix $A = (A_{ll'})_{l,l'=1}^n$ by

$$A_{ll'} = I_{w_{y_l}}[U_{y_{l'}}] := \int_{\partial\Omega} (u - U_{y_{l'}})(x) \frac{\partial w_{y_l}}{\partial \nu}(x)\, d\sigma(x) \,.$$

  - Small-volume expansion $\Rightarrow$

$$A_{ll'} \approx - \sum_{p=1}^{P} \frac{2(k_p - 1)|D_p|}{k_p + 1} \nabla U_{y_{l'}}(z_p) \cdot \nabla U_{y_l}(z_p) \,.$$

  - For $j = 1, 2$, introduce

$$g^{(j)}(z^S) = \Big(e_j \cdot \nabla U_{y_1}(z^S), \ldots, e_j \cdot \nabla U_{y_n}(z^S) \Big)^T, \quad z^S \in \Omega \,,$$

    $\{e_1, e_2\}$: orthonormal basis of $\mathbb{R}^2$.

Scale-Separation techniques

- MUSIC characterization (in terms of the range of the matrix A): There exists $n_0 > dP$ s.t. for any $n > n_0$

$$g^{(j)}(z^S) \in \mathrm{Range}(A) \text{ for } j = 1, 2 \text{ iff } z^S \in \{z_1, \ldots, z_P\}\ .$$

Scale-Separation techniques

- $\Pi_{\mathrm{noise}} = I - \Pi$, where Π: orthogonal projection onto the range of A.
- Given any point $z^S \in \Omega$, form the vector $g^{(j)}(z^S)$.
- Form an image of the anomalies by plotting, at each point z^S, the cost function

$$\mathcal{I}_{\mathrm{MU}}(z^S) = \frac{1}{\sqrt{||\Pi_{\mathrm{noise}}[g^{(1)}](z^S)||^2 + ||\Pi_{\mathrm{noise}}[g^{(2)}](z^S)||^2}} .$$

- Resulting plot will have large peaks at the locations of the anomalies.
- Once one locates the inclusions, $|D_p|(k_p - 1)/(k_p + 1)$, $p = 1, \ldots, P$: estimated from the significant singular values of A.

Scale-Separation techniques

- Direct imaging algorithms for the Helmholtz equation
- w: smooth function s.t. $(\Delta + k_0^2)w = 0$ in Ω.
- Weighted boundary measurements $I_w[U,\omega]$:

$$I_w[U,\omega] := \int_{\partial\Omega} (u - U)(x) \frac{\partial w}{\partial \nu}(x)\, d\sigma(x).$$

- Small-volume expansion $\Rightarrow$

$$I_w[U,\omega] \approx -\delta^d \Big(\nabla U(z) \cdot M(\lambda, B) \nabla w(z) + k_0^2 (\frac{\varepsilon_\star}{\varepsilon_0} - 1)|B| U(z) w(z) \Big).$$

Scale-Separation techniques

- P well-separated anomalies $D_p = z_p + \delta B_p$, $p = 1, \ldots, P$. Permeability and permittivity of D_p: μ_p and ε_p, respectively.
- B_p: disks $\Rightarrow$

$$I_w[U, \omega] \approx -\sum_{p=1}^{P} |D_p| \Big(2 \frac{\mu_0 - \mu_p}{\mu_0 + \mu_p} \nabla U(z_p) \cdot \nabla w(z_p) + k_0^2 (\frac{\varepsilon_p}{\varepsilon_0} - 1) U(z_p) w(z_p) \Big) .$$

Scale-Separation techniques

- MUSIC-type Algorithm:
  - $(\theta_1, \ldots, \theta_n)$: n unit vectors in $\mathbb{R}^d$.
  - $\theta \in \{\theta_1, \ldots, \theta_n\}$, $U(x) = e^{ik_0 \theta \cdot x}$.
  - $w(x) = e^{-ik_0 \theta' \cdot x}$ for $\theta' \in \{\theta_1, \ldots, \theta_n\}$.
  - $\Rightarrow$

$$I_w[U, \omega] \approx -\sum_{p=1}^{P} |D_p| k_0^2 \Big(2 \frac{\mu_0 - \mu_p}{\mu_0 + \mu_p} \theta \cdot \theta' + \frac{\varepsilon_p}{\varepsilon_0} - 1 \Big) e^{ik_0 (\theta - \theta') \cdot z_p} .$$

  - Response matrix $A = (A_{ll'})_{l,l'=1}^{n} \in \mathbb{C}^{n \times n}$:

$$A_{ll'} := I_{w_{l'}}[U_l, \omega] ,$$

$$U_l(x) = e^{ik_0 \theta_l \cdot x}, w_l(x) = e^{-ik_0 \theta_l \cdot x}, l = 1, \ldots, n.$$

Scale-Separation techniques

- For $l, l' = 1, \ldots, n$,

$$A_{ll'} \approx -\sum_{p=1}^{P} |D_p| k_0^2 \left(2 \frac{\mu_0 - \mu_p}{\mu_0 + \mu_p} \theta_l \cdot \theta_{l'} + \frac{\varepsilon_p}{\varepsilon_0} - 1 \right) e^{i k_0 (\theta_l - \theta_{l'}) \cdot z_p} .$$

- Introduce the n-dimensional vector fields $g^{(j)}(z^S)$, for $z^S \in \Omega$ and $j = 1, \ldots, d+1$,

$$g^{(j)}(z^S) = \frac{1}{\sqrt{n}} \left(e_j \cdot \theta_1 e^{i k_0 \theta_1 \cdot z^S}, \ldots, e_j \cdot \theta_n e^{i k_0 \theta_n \cdot z^S} \right)^T, \quad j = 1, \ldots, d ,$$

and

$$g^{(d+1)}(z^S) = \frac{1}{\sqrt{n}} \left(e^{i k_0 \theta_1 \cdot z^S}, \ldots, e^{i k_0 \theta_n \cdot z^S} \right)^T ,$$

$\{e_1, \ldots, e_d\}$: orthonormal basis of $\mathbb{R}^d$.

Scale-Separation techniques

- $g(z^S)$: $n \times d$ matrix whose columns are $g^{(1)}(z^S), \ldots, g^{(d)}(z^S)$.
-

$$A \approx -n \sum_{p=1}^{P} |D_p| k_0^2 \Big(2 \frac{\mu_0 - \mu_p}{\mu_0 + \mu_p} g(z_p) \overline{g(z_p)}^T + (\frac{\varepsilon_p}{\varepsilon_0} - 1) g^{(d+1)}(z_p) \overline{g^{(d+1)}(z_p)}^T \Big) \, .$$

- $\Pi_{\rm noise} = I - \Pi$, Π: orthogonal projection onto the range of A.
- MUSIC-type imaging functional:

$$\mathcal{I}_{\rm MU}(z^S, \omega) := \Big(\sum_{j=1}^{d+1} \| \Pi_{\rm noise}[g^{(j)}](z^S) \|^2 \Big)^{-1/2} .$$

- Large peaks only at the locations of the anomalies.

Scale-Separation techniques

- Backpropagation-type algorithms:
  - $(\theta_1, \ldots, \theta_n)$: n unit vectors in $\mathbb{R}^d$.
  - Backpropagation-type imaging functional at a single frequency ω:

$$\mathcal{I}_{\mathrm{BP}}(z^S, \omega) := \frac{1}{n} \sum_{l=1}^{n} e^{-2ik_0\theta_l \cdot z^S} I_{w_l}[U_l, \omega] \,;$$

    $U_l(x) = w_l(x) = e^{ik_0\theta_l \cdot x}$, $l = 1, \ldots, n$.
  - $(\theta_1, \ldots, \theta_n)$: equidistant points on the unit sphere S^{d-1}.

Scale-Separation techniques

- For sufficiently large n,

$$\frac{1}{n}\sum_{l=1}^{n} e^{ik_0\theta_l\cdot x} \approx -4\left(\frac{\pi}{k_0}\right)^{d-2} \Im\left\{\Gamma_{k_0}(x)\right\} = \begin{cases} \mathrm{sinc}(k_0|x|), & d=3\,, \\ J_0(k_0|x|), & d=2\,; \end{cases}$$

  $\mathrm{sinc}(s) = \sin(s)/s$: sinc function; J_0: Bessel function of the first kind and of order zero.

- Resolution limit:

$$\mathcal{I}_{\mathrm{BP}}(z^S,\omega) \approx -\sum_{p=1}^{P} |D_p| k_0^2 \left(2\frac{\mu_0-\mu_p}{\mu_0+\mu_p} + \left(\frac{\varepsilon_p}{\varepsilon_0} - 1\right)\right) \times \begin{cases} \mathrm{sinc}(2k_0|z^S - z_p|) & d=3, \\ J_0(2k_0|z^S - z_p|) & d=2\,. \end{cases}$$

Scale-Separation techniques

- $\mathcal{I}_{\mathrm{BP}}$ uses only the diagonal terms of the response matrix A.
- Use the whole matrix $\Rightarrow$ Kirchhoff migration functional:

$$\mathcal{I}_{\mathrm{KM}}(z^S, \omega) = \sum_{j=1}^{d+1} \overline{g^{(j)}(z^S)} \cdot A g^{(j)}(z^S) \ .$$

- $P = 1$; $\mu_\star = \mu_0$,

$$A = -n|D|k_0^2 (\frac{\varepsilon_\star}{\varepsilon_0} - 1) g^{(d+1)}(z) \overline{g^{(d+1)}(z)}^T$$

- $\mathcal{I}_{\mathrm{MU}}$: nonlinear function of $\mathcal{I}_{\mathrm{KM}}$

$$\mathcal{I}_{\mathrm{KM}}(z^S, \omega) = -n|D|k_0^2 (\frac{\varepsilon_\star}{\varepsilon_0} - 1) \left(1 - \mathcal{I}_{\mathrm{MU}}^{-2}(z^S, \omega) \right) \ .$$