Lecture 5: Photo-acoustic imaging

Habib Ammari

Department of Mathematics, ETH Zürich

Photo-acoustic imaging

- Photoacoustic imaging:
  - Optical energy absorption causes thermoelastic expansion of the tissue;
  - $\Rightarrow$ propagation of a pressure wave.
  - Acoustic signal measured by transducers distributed on the boundary of the object;
  - Imaging optical properties of the object.
- Major contribution of photoacoustic imaging: provide images of optical contrasts (based on the optical absorption) with the resolution of ultrasound.

Photo-acoustic imaging

- Acoustically homogeneous medium with the same acoustic properties as the free space:
  - The boundary of the object plays no role;
  - The optical properties of the medium can be extracted from measurements of the pressure wave by inverting a spherical or a circular mean Radon transform.
- The free space assumption does not hold:
  - Brain imaging: the skull plays an important acoustic role.
  - If a boundary condition has to be imposed on the pressure field, then an explicit inversion formula no longer exists.
  - Use of a geometric-control approach.
- Take into account the acoustic attenuation:
  - Use a frequency power-law model for the attenuation;
  - Compensate the effect of acoustic attenuation.

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- Photoacoustic imaging problem in free space:
    - Recover the absorbing energy density from limited-view data.
    - Correct the effect of acoustic attenuation:
        - SVD approach;
        - Technique based on the stationary phase theorem.
- Correct the effect of imposed boundary conditions.
- Quantitative photoacoustic imaging approach in the framework of small absorbers.

Photo-acoustic imaging

- Mathematical formulation.
- Acoustically homogeneous medium $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$; irradiated by optical laser pulse.
- Photoacoustic effect:

$$\frac{\partial^2 p}{\partial t^2}(x,t) - c_0^2 \Delta p(x,t) = \gamma \frac{\partial H}{\partial t}(x,t), \quad x \in \Omega, t \in \mathbb{R};$$

- c_0: acoustic speed; γ: dimensionless Grüneisen coefficient ; H: heat source function (absorbed energy per unit time per unit volume).
- γ: a measure of the conversion efficiency of heat energy to pressure.
- $D_l, l = 1, \ldots, L,$: absorbing domains inside the nonabsorbing background Ω.
- Stress-confinement assumption $\Rightarrow$ the source term modeled as

$$\gamma H(x,t) = \delta_0(t) \sum_{l=1}^{L} \chi(D_l) A_l(x),$$

δ_0: Dirac distribution at 0.

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- Pressure p satisfies

$$\frac{\partial^2 p}{\partial t^2}(x,t) - c_0^2 \Delta p(x,t) = 0 \quad x \in \Omega,\ t \in (0,T)$$

for some final observation time T, the initial conditions

$$p|_{t=0} = p_0 = \sum_{l=1}^{L} \chi(D_l) A_l(x) \quad \text{and} \quad \partial_t p|_{t=0} = 0\,,$$

and either the Dirichlet or the Neumann boundary condition (if Ω: bounded)

$$p = 0 \quad \text{or} \quad \frac{\partial p}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T)\,.$$

- Neumann boundary condition: tissue/water interface; Dirichlet boundary condition: tissue/air interface.

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- Inverse problem in photoacoustic imaging: determine the supports of nonzero optical absorption, $D_l \Subset \Omega, l = 1, \ldots, L$, and the absorbed optical energy density times the Grüneisen coefficient, $A(x) = \sum_{l=1}^{L} A_l(x)\chi(D_l)$, from boundary measurements of the pressure on $\partial\Omega$ (if Ω: bounded) or measurements on the boundary of a bounded domain if $\Omega = \mathbb{R}^d$.
- Final observation time T s.t.

$$T > \operatorname{diam}(\Omega)/c_0 \, .$$

- Inverse source problem.

Photo-acoustic imaging

- Density $A(x)$: related to the optical absorption coefficient distribution $\mu_a(x) = \sum_{l=1}^{L} \mu_l(x)\chi(D_l)$ by

$$A(x) = \gamma \mu_a(x)\Phi(x);$$

- Φ: light fluence

$$\Big(\mu_a - \frac{1}{3}\nabla \cdot \frac{1}{\mu_a + \mu_s}\nabla\Big)\Phi = 0 \quad \text{in } \Omega$$

with the boundary condition

$$\frac{\partial \Phi}{\partial \nu} + l\Phi = g \quad \text{on } \partial\Omega\,.$$

- μ_s: reduced scattering coefficient; g: light source; $1/l$: extrapolation length.
- Quantitative photoacoustic imaging: A: nonlinear function of μ_a.

Photo-acoustic imaging

- Photoacoustic imaging in free space: Full-view setting
- $c_0 = 1$; Support of p_0: contained in the unit disk Ω; $\partial\Omega$: boundary of Ω.
- Reconstruct p_0 from the measurements $g(y,t) = p(y,t)$ on $\partial\Omega \times (0,T)$.
- Filtered backprojection formula:

$$p_0(x) = \mathcal{R}^* \mathcal{B} \mathcal{R}[p_0](x)\,;$$

- $\mathcal{R}$: spherical mean Radon transform

$$\mathcal{R}[f](x,s) = \frac{1}{2\pi} \int_{\partial\Omega} f(x + s\xi)\, d\sigma(\xi), \quad (x,s) \in \partial\Omega \times \mathbb{R}^+\,;$$

- For $g : \partial\Omega \times \mathbb{R}^+ \to \mathbb{R}$, Backprojection operator $\mathcal{R}^*$:

$$\mathcal{R}^*[g](x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{g(y, |x-y|)}{|x-y|}\, d\sigma(y)\,;$$

- Filter $\mathcal{B}$:

$$\mathcal{B}[g](x,t) = \int_0^2 \frac{\partial^2 g}{\partial s^2}(x,s) \log(|s^2 - t^2|)\, ds\,.$$

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- $\mathcal{B}$: symmetric and positive.
- Wave operator $\mathcal{W}$: for $g : \partial\Omega \times \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{W}[g](y,t) = \frac{2}{\pi} \int_0^t \frac{g(y,s)}{\sqrt{t^2 - s^2}}\, ds$$

- Kirchhoff formula $\Rightarrow$

$$\mathcal{R}[p_0](y,t) = \mathcal{W}[p](y,t)\,,$$

- Filtered backprojection formula $\to$ initial data p_0 from measurements of the pressure p on $\partial\Omega \times \mathbb{R}^+$.

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- Time-reversal imaging: applied in order to reconstruct the initial data $p_0(x)$ from measurements of $g(y,t) = \mathcal{W}[p](y,t)$ for $(y,t) \in \partial\Omega \times (0,T)$.
- v: solution to

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}(x,t) - \Delta_x v(x,t) = 0, & (x,t) \in \Omega \times [0,T]\,, \\ v(x,0) = \dfrac{\partial v}{\partial t}(x,0) = 0, & x \in \Omega\,, \\ v(x,t) = \dfrac{1}{2} g(x, T-t), & (x,t) \in \partial\Omega \times [0,T]\,. \end{cases}$$

- Time-reversal imaging function: $\mathcal{I}_{\mathrm{TR}}^{(1)}(x) := v(x,T), \quad x \in \Omega\,,$

$$\mathcal{I}_{\mathrm{T}R}^{(1)}(x) = \frac{1}{2} p_0(x), \quad x \in \Omega\,.$$

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- Time-dependent fundamental solution:

$$\Gamma(x,y,s,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \Gamma_\omega(x,y)\exp(-i\omega(t-s))d\omega\,;$$

- Γ_ω: fundamental solution to the Helmholtz equation $(\Delta+\omega^2)$ in $\mathbb{R}^d$ subject to the outgoing radiation condition.
- Γ solution to

$$\begin{cases} \dfrac{\partial^2\Gamma}{\partial t^2}(x,y,s,t) - \Delta_y\Gamma(x,y,s,t) = -\delta_x(y)\delta_s(t), & (y,t)\in\mathbb{R}^d\times\mathbb{R}\,, \\ \Gamma(x,y,s,t) = \dfrac{\partial\Gamma}{\partial t}(x,y,s,t) = 0, & y\in\mathbb{R}^d, \quad t<s\,. \end{cases}$$

- $$g(y,t) = -\int_\Omega \frac{\partial\Gamma}{\partial t}(z,y,0,t)p_0(z)\,dz, \quad y\in\partial\Omega, \quad t\in[0,T]\,.$$

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- Retarded Dirichlet Green function:

$$\begin{cases} \dfrac{\partial^2 G^{\mathrm{c}}}{\partial t^2}(x,y,s,t) - \Delta_y G^{\mathrm{c}}(x,y,s,t) = -\delta_x(y)\delta_s(t), & (y,t) \in \Omega \times \mathbb{R}\,, \\ G^{\mathrm{c}}(x,y,s,t) = 0, \quad \dfrac{\partial G^{\mathrm{c}}}{\partial t}(x,y,s,t) = 0, & y \in \Omega, \quad t < s\,, \\ G^{\mathrm{c}}(x,y,s,t) = 0, & (y,t) \in \partial\Omega \times \mathbb{R}\,. \end{cases}$$

- Advanced Dirichlet Green function:

$$\begin{cases} \dfrac{\partial^2 G^{\mathrm{a}}}{\partial t^2}(x,y,s,t) - \Delta_y G^{\mathrm{a}}(x,y,s,t) = -\delta_x(y)\delta_s(t), & (y,t) \in \Omega \times \mathbb{R}\,, \\ G^{\mathrm{a}}(x,y,s,t) = 0, \quad \dfrac{\partial G^{\mathrm{a}}}{\partial t}(x,y,s,t) = 0, & y \in \Omega, \quad t > s\,, \\ G^{\mathrm{a}}(x,y,s,t) = 0, & (y,t) \in \partial\Omega \times \mathbb{R}\,. \end{cases}$$

- Time-reversibility of the wave equation:

$$G^{\mathrm{a}}(x,y,s,t) = G^{\mathrm{c}}(x,y,s,2s-t) = G^{\mathrm{c}}(x,y,t,s)\,.$$

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- Symmetric Dirichlet Green function:

$$G(x, y, s, t) = \frac{1}{2}\Big(G^{\mathrm{a}}(x, y, s, t)\chi((-\infty, s))(t) + G^{\mathrm{c}}(x, y, s, t)\chi((s, \infty))(t)\Big) .$$

- G: contains both the causal and anticausal Green functions,

$$\begin{cases} \dfrac{\partial^2 G}{\partial t^2}(x, y, s, t) - \Delta_y G(x, y, s, t) = -\delta_x(y)\delta_s(t), & (y, t) \in \Omega \times \mathbb{R}, \\ G(x, y, s, t) = 0, & (y, t) \in \partial\Omega \times \mathbb{R} . \end{cases}$$

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- $\mathcal{I}_{\mathrm{TR}}^{(1)}$:

$$\mathcal{I}_{\mathrm{TR}}^{(1)}(x) = v(x,T) = \frac{1}{2}\int_0^T \int_{\partial\Omega} \frac{\partial G^{\mathrm{c}}(x,y,s,T)}{\partial \nu_y} g(y, T-s)\, d\sigma(y)\, ds\ .$$

- $\Rightarrow$

$$\begin{aligned}
\mathcal{I}_{\mathrm{TR}}^{(1)}(x) &= \frac{1}{2}\int_0^T \int_{\partial\Omega} \frac{\partial G^{\mathrm{a}}(x,y,T,s)}{\partial \nu_y} g(y, T-s)\, d\sigma(y)\, ds \\
&= \int_0^T \int_{\partial\Omega} \frac{\partial G(x,y,T,s)}{\partial \nu_y} g(y, T-s)\, d\sigma(y)\, ds \\
&= \int_0^T \int_{\partial\Omega} \frac{\partial G(x,y,0,t)}{\partial \nu_y} g(y, t)\, d\sigma(y)\, dt\ ,
\end{aligned}$$

  because G is even.

- $g(x,t) = 0$ for $t \geq T$ and for $t \leq 0 \Rightarrow$

$$\mathcal{I}_{\mathrm{TR}}^{(1)}(x) = \int_{\mathbb{R}} \int_{\partial\Omega} \frac{\partial G(x,y,0,t)}{\partial \nu_y} g(y, t)\, d\sigma(y)\, dt\ .$$

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- $\mathcal{I}_{\mathrm{TR}}^{(1)}(x)$: gives a perfect image of $p_0(x)$.
- $$\mathcal{I}_{\mathrm{TR}}^{(1)}(x) = -\frac{i}{2\pi} \int_\Omega f(z) \int_{\mathbb{R}} \omega \int_{\partial\Omega} \frac{\partial G_\omega}{\partial \nu_y}(x,y)\overline{\Gamma_\omega}(z,y) d\sigma(y)\, d\omega\, dz\ .$$
- Integration by parts over $\Omega \Rightarrow$

$$\int_{\partial\Omega} \frac{\partial G_\omega}{\partial \nu_y}(x,y)\overline{\Gamma_\omega}(z,y) d\sigma(y) = \overline{\Gamma_\omega}(z,x) - G_\omega(x,z) = \overline{\Gamma_\omega}(x,z) - G_\omega(x,z)\ ,$$

Photo-acoustic imaging

- G_ω: real-valued ($t \to G(x, y, 0, t)$: real and even) $\Rightarrow$

$$\Im \int_{\partial\Omega} \frac{\partial G_\omega}{\partial \nu_y}(x, y)\overline{\Gamma_\omega}(z, y) d\sigma(y) = \Im \left\{ \overline{\Gamma_\omega}(x, z) \right\} = -\Im \left\{ \Gamma_\omega(x, z) \right\} \; .$$

- $$\mathcal{I}_{\mathrm{TR}}^{(1)}(x) = -\frac{1}{2\pi} \int_\Omega p_0(z) \int_{\mathbb{R}} \omega \Im \left\{ \Gamma_\omega(x, z) \right\} \, d\omega \, dz \; .$$

- From

$$\frac{1}{\pi} \int_{\mathbb{R}} \omega \Im \left\{ \Gamma_\omega(x, z) \right\} d\omega = -\delta_z(x) \; ,$$

- $\Rightarrow$

$$2\mathcal{I}_{\mathrm{TR}}^{(1)}(x) = p_0(x) \; .$$

Photo-acoustic imaging

- Modified time-reversal imaging
- Use "free boundary conditions":

$$v_s(x,t) = -\int_{\partial\Omega} \frac{\partial\Gamma}{\partial t}(x,y,s,t) g(y, T-s)\, d\sigma(y)\ .$$

- For $s \in [0, T]$,

$$\begin{cases} \dfrac{\partial^2 v_s}{\partial t^2}(x,t) - \Delta_x v_s(x,t) = \dfrac{d\delta_s}{dt}(t) g(x, T-s)\delta_{\partial\Omega}(x), & (x,t) \in \mathbb{R}^d \times \mathbb{R}\ , \\ v_s(x,t) = 0, \quad \dfrac{\partial v_s}{\partial t}(x,t) = 0, & x \in \mathbb{R}^d, \quad t < s\ . \end{cases}$$

$\delta_{\partial\Omega}$: surface Dirac measure on $\partial\Omega$.

- Modified time-reversal imaging functional:

$$\mathcal{I}_{\mathrm{TR}}^{(2)}(x) = \int_0^T v_s(x,T) ds, \quad x \in \Omega\ .$$

Photo-acoustic imaging

- $\mathcal{I}_{\mathrm{TR}}^{(2)}$: approximation of $\mathcal{I}_{\mathrm{TR}}^{(1)}$.
- $g_\omega(y) = \int g(y,s) e^{i\omega s} ds$: Fourier transform of g,

$$g_\omega(y) = i\omega \int_\Omega \Gamma_\omega(z,y) p_0(z)\, dz \ .$$

- Parseval's relation $\Rightarrow$

$$\begin{aligned}
\mathcal{I}_{\mathrm{TR}}^{(2)}(x) &= -\int_0^T \int_{\partial\Omega} \frac{\partial \Gamma}{\partial t}(x,y,0,t) g(y,t) d\sigma(y) dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\partial\Omega} i\omega \Gamma_\omega(x,y) \overline{g}_\omega(y) d\sigma(y) d\omega \ , \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^d} p_0(z) \int_{\mathbb{R}} \int_{\partial\Omega} \omega^2 \Gamma_\omega(x,y) \overline{\Gamma}_\omega(z,y) d\sigma(y) d\omega dz \ .
\end{aligned}$$

Photo-acoustic imaging

- Helmholtz-Kirchhoff identity:

$$\int_{\partial\Omega} \Gamma_\omega(x,y)\overline{\Gamma}_\omega(z,y)d\sigma(y) \approx -\frac{1}{\omega}\Im\left\{\Gamma_\omega(x,z)\right\} \ ,$$

- $\Rightarrow$

$$2\mathcal{I}_{\mathrm{TR}}^{(2)}(x) \approx -\frac{1}{\pi}\int_{\mathbb{R}^d} p_0(z)\int_{\mathbb{R}} \omega\Im\left\{\Gamma_\omega(x,z)\right\} d\omega dz \ \approx p_0(x) \ .$$

Photo-acoustic imaging

- Limited-view setting.
- Disposal data on $S \times (0, T)$, where $S \subset \partial\Omega$.
- Restriction: not stable enough to give a correct reconstruction of p_0,

$$p_0(x) \simeq \frac{1}{2\pi} \int_\Gamma \int_0^2 \left[\frac{d^2}{dt^2} \mathcal{R}[p_0](y,t) \right] \log |t^2 - |y-x|^2| \, dt \, d\sigma(y) \, .$$

- Inverse problem becomes severely ill-posed: regularized.
- Total variation regularization: regularized minimization problem,

$$\min_{p_0} J_\gamma[f] := \frac{1}{2} \|\mathcal{B}^{1/2} \left[\mathcal{R}[p_0] - g \right] \|^2_{L^2(S\times(0,2))} + \gamma \|\nabla p_0\|_{L^1(\Omega)} \, ;$$

γ: regularization parameter.

Photo-acoustic imaging

- Compensation of the effect of acoustic attenuation
- Attenuating medium: p_a solution to

$$\frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2}(x,t) - \Delta p_a(x,t) - L(t) \star p_a(x,t) = \frac{1}{c_0^2}\frac{d}{dt}\delta_0(t) p_0(x),$$

- L:

$$L(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(K^2(\omega) - \frac{\omega^2}{c_0^2}\right) e^{-i\omega t} d\omega .$$

- Causal models for $K(\omega)$:
  - Power-law model

$$K(\omega) = \frac{\omega}{c(\omega)} + ia|\omega|^\zeta ,$$

  ω: frequency, $c(\omega)$: frequency dependent phase velocity and $1 \leq \zeta \leq 2$: power of the attenuation coefficient.
  - Thermoviscous model:

$$K(\omega) = \frac{\omega}{c_0\sqrt{1 - ia\omega c_0}}.$$

Photo-acoustic imaging

- Strategy:
  - Estimate the solution $p(y, t)$ of the non-attenuated wave equation

    $$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}(x,t) - \Delta p(x,t) = \frac{1}{c_0^2}\frac{d}{dt}\delta_0(t)p_0(x)\,,$$

    from $p_a(y, t)$ for all $(y, t) \in \partial\Omega \times \mathbb{R}^+$.
  - Apply the inverse formula for the spherical mean Radon transform to reconstruct p_0 from the non-attenuated data.

Photo-acoustic imaging

- Relationship between p and p_a
- Fourier transforms $\hat{p} = \mathcal{F}_{\mathrm{t}}[p]$ and $\hat{p}_a = \mathcal{F}_{\mathrm{t}}[p_a]$ satisfy

$$\left(\Delta + (\frac{\omega}{c_0})^2\right) \hat{p}(x,\omega) = \frac{i\omega}{\sqrt{2\pi} c_0^2} p_0(x)$$

  and

$$\left(\Delta + K(\omega)^2\right) \hat{p}_a(x,\omega) = \frac{i\omega}{\sqrt{2\pi} c_0^2} p_0(x) \,.$$

- $\Rightarrow$

$$\hat{p}(x, c_0 K(\omega)) = \frac{c_0 K(\omega)}{\omega} \hat{p}_a(x,\omega).$$

- Estimate p from p_a using the relationship $p_a = \mathcal{L}[p]$,

$$\mathcal{L}[\phi](s) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\omega}{c_0 K(\omega)} e^{-i\omega s} \int_0^\infty \phi(t) e^{i c_0 K(\omega) t} \, dt \, d\omega \,.$$

- $\mathcal{L}$: not well conditioned.
- Regularized inverse of $\mathcal{L}$ via a singular value decomposition.
- Asymptotic behavior of $\mathcal{L}$ as the attenuation coefficient $a \to 0$.

Photo-acoustic imaging

- SVD Approach:
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$$\mathcal{L}[\phi] = \sum_l \sigma_l (\widetilde{\psi}_l, \phi) \psi_l,$$

$(\widetilde{\psi}_l)$ and (ψ_l): two orthonormal bases of $L^2((0,T))$ and σ_l: positives eigenvalues s.t.

$$\begin{cases} \mathcal{L}^*[\phi] & = \sum_l \sigma_l (\psi_l, \phi) \widetilde{\psi}_l \,, \\ \mathcal{L}^*\mathcal{L}[\phi] & = \sum_l \sigma_l^2 (\widetilde{\psi}_l, \phi) \widetilde{\psi}_l \,, \\ \mathcal{L}\mathcal{L}^*[\phi] & = \sum_l \sigma_l^2 (\psi_l, \phi) \psi_l \,. \end{cases}$$

- Approximate inverse of $\mathcal{L}$:

$$\mathcal{L}_{1,\gamma}^{-1}[\phi] = \sum_l \frac{\sigma_l}{\sigma_l^2 + \gamma^2} (\psi_l, \phi) \widetilde{\psi}_l \,,$$

$\gamma > 0$: regularization parameter.

Photo-acoustic imaging

- Coefficient of attenuation a: very small.
- Approximation of $\mathcal{L}$ and $\mathcal{L}^{-1}$ as $a \to 0$:

$$\mathcal{L}_j[\phi] = \mathcal{L}[\phi] + o(a^{j+1}) \quad \text{and} \quad \mathcal{L}_{2,j}^{-1}[\phi] = \mathcal{L}^{-1}[\phi] + o(a^{j+1}),$$

  j: order of approximation.
- Thermoviscous model:

$$\mathcal{L}[\phi](s) \simeq \frac{1}{2\pi} \int_0^\infty \phi(t) \int_{\mathbb{R}} \left(1 - i\frac{ac_0}{2}\omega\right) e^{-\frac{1}{2}c_0 a\omega^2 t} e^{i\omega(t-s)}\, d\omega\, dt\,.$$

- From

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}c_0 a\omega^2 t} e^{i\omega(t-s)} d\omega = \frac{1}{\sqrt{c_0 at}} e^{-\frac{1}{2}\frac{(s-t)^2}{c_0 at}}\,,$$

$$\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \omega e^{-\frac{1}{2}c_0 a\omega^2 t} e^{i\omega(t-s)} d\omega = -\partial_s \left(\frac{1}{\sqrt{c_0 at}} e^{-\frac{1}{2}\frac{(s-t)^2}{c_0 at}}\right)\,,$$

- $\Rightarrow$

$$\mathcal{L}[\phi] \simeq \left(1 + \frac{ac_0}{2}\partial_s\right) \left(\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \phi(t) \frac{1}{\sqrt{c_0 at}} e^{-\frac{1}{2}\frac{(s-t)^2}{c_0 at}} dt \right)\,.$$

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- Introduce

$$\widetilde{\mathcal{L}}[\phi] := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \phi(t) \frac{1}{\sqrt{c_0 a t}} e^{-\frac{1}{2}\frac{(s-t)^2}{c_0 a t}} dt \,.$$

- Quadratic phase and a: small $\Rightarrow$ (stationary phase theorem)

$$\widetilde{\mathcal{L}}[\phi](s) = \sum_{l=0}^{j} \frac{(c_0 a)^l}{2^l l!} D_l[\phi](s) + o(a^j)\,,$$

$D_l[\phi](s) = (t^l \phi(t))^{(2l)}(s)$.

- Approximation of order j of $\widetilde{\mathcal{L}}^{-1}$:

$$\widetilde{\mathcal{L}}_0^{-1}[\psi] = \psi, \quad \widetilde{\mathcal{L}}_1^{-1}[\psi] = \psi - \frac{ac_0}{2}(t\psi)''\,,$$

- $\Rightarrow$

$$\mathcal{L}[\phi] = \phi + \frac{ac_0}{2}(t\phi')' + o(a) \quad \text{and } \mathcal{L}_{2,1}^{-1}[\psi] = \psi - \frac{ac_0}{2}(t\psi')'\,.$$

Photo-acoustic imaging

- Photoacoustic imaging of small absorbers with imposed boundary conditions on the pressure
- Ω: bounded smooth domain. Consider the wave equation in Ω with the Dirichlet (resp. the Neumann) imposed boundary conditions on $\partial\Omega \times (0, T)$.
- Reconstruct $p_0(x)$ from the measurements of $\dfrac{\partial p}{\partial \nu}(x, t)$ (resp. $p(x, t)$) on the boundary $\partial\Omega \times (0, T)$.
- Identify (well-separated) small absorbing regions from boundary measurements: $D_l, l = 1, \ldots, L$, absorbing domains inside the nonabsorbing background Ω.
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$$D_l = z_l + \delta B_l,$$
  z_l: "center" of D_l, B_l contains the origin and plays a role of a reference domain, and δ: common order of magnitude of the diameters of the D_l.

Photo-acoustic imaging

- Probe wave:

$$\frac{1}{c_0^2}\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 \quad \text{in } \mathbb{R}^d \times (0,T)\,,$$

with the final conditions

$$v|_{t=T} = \frac{\partial v}{\partial t}\Big|_{t=T} = 0 \quad \text{in } \Omega\,.$$

- Integration by parts ⇒

$$\int_0^T \int_{\partial\Omega} \frac{\partial p}{\partial \nu}(x,t) v(x,t)\, d\sigma(x)\, dt = \frac{1}{c_0^2} \sum_{l=1}^{L} \int_{D_l} A_l(x) \partial_t v(x,0) dx\,.$$

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- Suppose $d = 3$. For $y \in \mathbb{R}^3 \setminus \overline{\Omega}$, consider the spherical wave emitted by a point source at y at time $-\tau$,

$$v_y(x,t;\tau) := \frac{\delta_0\left(t+\tau-\frac{|x-y|}{c_0}\right)}{4\pi|x-y|} \quad \text{in } \Omega\times(0,T),$$

  $\tau > \frac{\text{dist}(y,\partial\Omega)}{c_0}$: parameter.

- Suppose that

$$A_l(x) = \sum_{|j|=0}^{N} \frac{1}{j!} a_j^{(l)} (x-z_l)^j .$$

-

$$\frac{1}{c_0^2}\sum_{l=1}^{L}\sum_{|j|=0}^{N}\frac{1}{j!}a_j^{(l)}\int_{D_l}(x-z_l)^j\partial_t v_y(x,0;\tau)dx$$

$$= \int_0^T\int_{\partial\Omega}\frac{\partial p}{\partial\nu}(x,t)v_y(x,t;\tau)\,d\sigma(x)\,dt\,.$$

Photo-acoustic imaging

- Determination of location: $(L = 1)$

$$\tau \mapsto \int_0^T \int_{\partial\Omega} \frac{\partial p}{\partial \nu}(x,t) v_y(x,t;\tau)\, d\sigma(x)\, dt$$

nonzero only on the interval $[\tau_a, \tau_e]$, $\tau_a = \text{dist}(y, D)/c_0$: the first τ for which the sphere of center y and radius τ hits D and τ_e: the last τ for which such sphere hits D.

Photo-acoustic imaging

- Imaging with limited-view data.
- Reconstruction of the location z from boundary measurements of $\frac{\partial p}{\partial \nu}$ on $S \times (0, T)$; $S \Subset \partial\Omega$.
- Use as weights particular background solutions constructed by the geometrical control method.

Photo-acoustic imaging

- Geometrical control of the wave equation:
- T and S: geometrically control Ω;
- $\beta \in \mathcal{C}_0^\infty(\Omega)$: cutoff function s.t. $\beta(x) \equiv 1$ in $\Omega' \Subset \Omega$, $z \in \Omega'$.
- For any function $w \in H^1(\Omega)$, there exists a unique $g_w(x,t) \in H_0^1(0,T;L^2(S))$ s.t. $v \in \mathcal{C}^0(0,T;L^2(\Omega)) \cap \mathcal{C}^1(0,T;H^{-1}(\Omega))$

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - c_0^2 \Delta v = 0 \quad \text{in } \Omega \times (0,T)\,, \\ v = 0 \quad \text{on } \partial\Omega \setminus \overline{S} \times (0,T)\,, \\ v = g_w \quad \text{on } S \times (0,T)\,, \\ v(x,0) = c_0^2 \beta(x) w(x), \quad \partial_t v(x,0) = 0 \quad \text{in } \Omega\,, \end{cases}$$

satisfies the final conditions

$$v\big|_{t=T} = \frac{\partial v}{\partial t}\Big|_{t=T} = 0 \quad \text{in } \Omega\,.$$

Photo-acoustic imaging

- Reconstruction procedure from measurements only on S:

$$\int_0^T \int_S \frac{\partial p}{\partial \nu}(x,t) v(x,t) d\sigma(x) dt = \int_\Omega p_0(x) \frac{\partial v}{\partial t}(x,0) dx \, .$$

- Choice of w:

$$w(x) := \frac{\delta_0 \left(\tau - \frac{|x-y|}{c_0} \right)}{4\pi |x-y|} .$$