Lecture 6: Quantitative thermo-acoustic imaging

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Quantitative thermo-acoustic imaging

- Quantitative thermoacoustic imaging:
  - Reconstruct the absorption coefficient from thermal energy measurements;
  - Given several data sets $\Rightarrow$ analytical reconstruction formula;
  - Formula involves derivatives of the given data $\Rightarrow$ unstable: small measurement noises may cause large errors;
  - Regularization technique $\Rightarrow$ good initial guess;
  - Optimal control approach.

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- u^k: solution of

$$\begin{cases} \Delta u^k + (k^2 + ikq)u^k = 0 & \text{in } \Omega\,, \\ u^k = g & \text{on } \partial\Omega\,. \end{cases}$$

- Thermoacoustic imaging problem: inverse problem of reconstructing the absorption coefficient q from thermoacoustic measurements $q|u^k|^2$ in Ω for $k \in (\underline{k}, \overline{k})$.
- $q|u^k|^2$ in Ω: heat energy due to the absorption distribution q.
- Heat energy generates an acoustic wave propagating inside the medium. Finding the initial data in the acoustic wave from boundary measurements $\rightarrow$ the heat energy distribution.
- Quantitative imaging: separate q from u^k.

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- Polarization procedure to obtain more data:
  - $g_1, g_2 \in L^2(\partial\Omega)$. For $j = 1, 2$, and $k \in (\underline{k}, \overline{k})$, u_j^k: solution of

$$\begin{cases} \Delta u_j^k + (k^2 + ikq) u_j^k = 0 \quad \text{in } \Omega\,, \\ u_j^k = g_j \quad \text{on } \partial\Omega\,. \end{cases}$$

  - Function:

$$E_2^k(x) = q(x) u_2^k(x) \overline{u_1^k(x)}, x \in \Omega$$

    can be evaluated from the knowledge of the data
    $q|u_1^k + u_2^k|^2$ and $q|iu_1^k + u_2^k|^2 \leftarrow g = g_1 + g_2$ and $g = ig_1 + g_2$.
  - Proof:

$$E_2^k = \frac{1}{2}(q|u_1^k + u_2^k|^2 - q|u_1^k|^2 - q|u_2^k|^2) + \frac{i}{2}(q|iu_1^k + u_2^k|^2 - q|u_1^k|^2 - q|u_2^k|^2).$$

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- u_j^k: solution corresponding to $(g_j)_{j=1}^{d+1} = (1, x_1, \ldots, x_d)$.
- Polarized data: $\mathcal{E}^k = (E_j^k)_{j=1}^{d+1} = (q u_j^k \overline{u}_1^k)_{j=1}^{d+1} \quad \forall\, k \in (\underline{k}, \overline{k})$.
- Proper set of measurements:
  - Assume that q: compactly supported in Ω and denote by Ω' its support.
  - $\Rightarrow$ There exist $N > 1$ pairwise disjoint open subsets $B_1, B_2, \ldots, B_N$ of Ω, and N frequencies $k_1, \ldots, k_N \in (\underline{k}, \overline{k})$ s.t. $\overline{\Omega'} \subset \cup_{j=1}^N \overline{B}_j \subset \Omega$ and, for any $n = 1, \ldots, N$,

    (i) $|u_1^{k_n}| > 0$ in B_n;

    (ii) The matrix $[u_j^{k_n}, \nabla u_j^{k_n}]_{1 \leq j \leq d+1}$ is invertible for all $x \in B_n$.

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- Proof:
  - Fix an arbitrary point $z \in \Omega$, assume that $u_1^k(z) = 0$ for all $k \in (\underline{k}, \overline{k})$.
  - $k \mapsto u_1^k(z)$: analytic $\Rightarrow u_1^k(z) = 0$ for all $k \in \mathbb{R} \Rightarrow u_1^0(z) = 0$.
  - $k = 0$ and $g = 1 \Rightarrow u_1^0(z) = 1$.
  - $\Rightarrow$ For all $z \in \Omega$, there is $k_z \in (\underline{k}, \overline{k})$ s.t. $u_1^{k_z}(z)$ does not vanish.
  - By the continuity of $u_1^{k_z}$, $|u_1^{k_z}| > 0$ in B_z, a small neighborhood of z in Ω.
  - $\overline{\Omega}$: compact $\Rightarrow$ one can extract $B_1, \ldots, B_N$ from $\{B_z : z \in \Omega\}$ s.t. $\overline{B}_1, \ldots, \overline{B}_N$ cover $\overline{\Omega}$, and hence (i) holds.
  - (ii) can be proved similarly using the differentiability of the determinant.
- $(g_j)_{j=1}^{d+1} \subset L^2(\partial\Omega)$: proper set of measurements on $(\underline{k}, \overline{k})$ iff the corresponding (u_j^k), $j = 1, \ldots, d+1$, $k \in (\underline{k}, \overline{k})$ satisfy (i) and (ii).

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- Exact formula:
- Proper set of measurements $(g_j)_{j=1}^{d+1}$.
- Fix $n \in \{1, \ldots, N\}$. In B_n, for $2 \leq j \leq d+1$ and $1 \leq l \leq N$,

$$u_j^{k_l} = \alpha_j^{k_l} u_1^{k_l}, \quad \alpha_j^{k_l} = \frac{E_j^{k_l}}{E_1^{k_l}}.$$

- Let $\beta^{k_l} = \Im(\overline{u}_1^{k_l} \nabla u_1^{k_l})$. Then

$$-\nabla \cdot \beta^{k_l} = k_l E_1^{k_l} \quad \text{in } B_n.$$

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- Proof:
  - $\varphi \in \mathcal{C}_c^\infty(B_n, \mathbb{R})$: arbitrary function; Use $\varphi u_1 \in H_0^1(B_n)$ as a test function in

$$-\Delta u_1^{k_l} = (k_l^2 + i k_l q) u_1^{k_l}$$

  - $\Rightarrow$

$$\int_\Omega \varphi |\nabla u_1^{k_l}|^2 dx + \int_\Omega \overline{u_1^{k_l}} \nabla u_1^{k_l} \cdot \nabla \varphi dx = \int_\Omega (k_l^2 + i k_l q) |u_1^{k_l}|^2 \varphi dx \,.$$

  - Taking the imaginary part $\Rightarrow$

$$-\int_\Omega \nabla \cdot (\Im \overline{u_1^{k_l}} \nabla u_1^{k_l}) \varphi dx = \int_\Omega k_l q |u_1^{k_l}|^2 \varphi dx = \int_\Omega k_l E_1^{k_l} \varphi dx \,.$$

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- For all $2 \leq j \leq d+1$ and $1 \leq l \leq N$,

$$\nabla \alpha_j^{k_l} \cdot \left(\nabla \log \frac{q}{E_1^{k_l}} - \frac{2iq\beta^{k_l}}{E_1^{k_l}} \right) = \Delta \alpha_j^{k_l} \quad \text{in } B_n.$$

- Proof: Fix $j \in \{2, \ldots, d+1\}$. $u_j^{k_l}$: solution of the Helmholtz equation $\Rightarrow$

$$\begin{aligned} (k_l^2 + ik_l q)\alpha_j^{k_l} u_1^{k_l} &= -\Delta\big(\alpha_j^{k_l} u_1^{k_l}\big) \\ &= (k_l^2 + ikq)\alpha_j^{k_l} u_1^{k_l} - u_1^{k_l} \Delta \alpha_j^{k_l} - 2\nabla u_1^{k_l} \cdot \nabla \alpha_j^{k_l}. \end{aligned}$$

- $\Rightarrow$

$$\begin{aligned} -E_1^{k_l} \Delta \alpha_j^{k_l} &= 2q\overline{u_1^{k_l}} \nabla u_1^{k_l} \cdot \nabla \alpha_j^{k_l} \\ &= q \left(\nabla |u_1^{k_l}|^2 + 2i\Im \overline{u_1^{k_l}} \nabla u_1^{k_l} \right) \cdot \nabla \alpha_j^{k_l}. \end{aligned}$$

- $\Rightarrow q\nabla |u_1^{k_l}|^2 \cdot \nabla \alpha_j^{k_l} = -E_1^{k_l} \Delta \alpha_j^{k_l} - 2iq\beta^{k_l} \cdot \nabla \alpha_j^{k_l}$.
- Differentiating $E_1^{k_l} = q|u_1^{k_l}|^2 \Rightarrow$

$$\nabla E_1^{k_l} = q\nabla |u_1^{k_l}|^2 + E_1^{k_l} \nabla \log q.$$

- $\Rightarrow (\nabla E_1^{k_l} - E_1^{k_l} \nabla \log q) \cdot \nabla \alpha_j^{k_l} = -E_1^{k_l} \Delta \alpha_j^{k_l} - 2iq\beta^{k_l} \cdot \nabla \alpha_j^{k_l}$.

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- The set $\{\nabla \alpha_j^{k_l}\}_{2\leq j\leq d+1}$: linearly independent for all $x \in \overline{\Omega}$.
- $d = 2$:

$$
\begin{aligned}
\det \begin{bmatrix} \nabla \alpha_2^{k_l} \\ \nabla \alpha_3^{k_l} \end{bmatrix} &= \frac{1}{(u_1^{k_l})^4} \det \begin{bmatrix} u_1^{k_l} \nabla u_2^{k_l} - u_2^{k_l} \nabla u_1^{k_l} \\ u_1^{k_l} \nabla u_3^{k_l} - u_3^{k_l} \nabla u_1^{k_l} \end{bmatrix} \\
&= \frac{1}{(u_1^{k_l})^4} \left(\det \begin{bmatrix} u_1^{k_l} \nabla u_2^{k_l} \\ u_1^{k_l} \nabla u_3^{k_l} - u_3^{k_l} \nabla u_1^{k_l} \end{bmatrix} \right. \\
&\qquad \left. - u_2^{k_l} \det \begin{bmatrix} \nabla u_1^{k_l} \\ u_1^{k_l} \nabla u_3^{k_l} - u_3^{k_l} \nabla u_1^{k_l} \end{bmatrix} \right) \\
&= \frac{1}{(u_1^{k_l})^3} \left(u_1^{k_l} \det \begin{bmatrix} \nabla u_2^{k_l} \\ \nabla u_3^{k_l} \end{bmatrix} + u_3^{k_l} \det \begin{bmatrix} \nabla u_1^{k_l} \\ \nabla u_2^{k_l} \end{bmatrix} \right. \\
&\qquad \left. - u_2^{k_l} \det \begin{bmatrix} \nabla u_1^{k_l} \\ \nabla u_3^{k_l} \end{bmatrix} \right) \\
&= \frac{1}{(u_1^{k_l})^3} \det \begin{bmatrix} u_1^{k_l} & \nabla u_1^{k_l} \\ u_2^{k_l} & \nabla u_2^{k_l} \\ u_3^{k_l} & \nabla u_3^{k_l} \end{bmatrix} \neq 0\,.
\end{aligned}
$$

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- $A^{k_l} = [\nabla \alpha_{j+1}^{k_l}]_{1\leq j\leq d}$: invertible in $B_n \Rightarrow$

$$\nabla \log \frac{q}{E_1^{k_l}} - \frac{2iq\beta^{k_l}}{E_1^{k_l}} = a^{k_l},$$

a^{k_l}: vector $({A^{k_l}}^* A^{k_l})^{-1}[{A^{k_l}}^*(\nabla \cdot A^{k_l})]$.

- Evaluate q: split $\Re$ and $\Im$ parts $\Rightarrow$

$$\nabla \log \frac{q}{E_1^{k_l}} = \frac{\nabla q}{q} - \nabla \log E_1^{k_l} = \Re(a^{k_l}), \quad \beta^{k_l} = -\frac{E_1^{k_l}\Im(a^{k_l})}{2q}.$$

- Differentiation $\Rightarrow$

$$\nabla \cdot \beta^{k_l} = \frac{E_1^{k_l}\Im(a^{k_l})\cdot \nabla q}{2q^2} - \frac{\nabla \cdot (E_1^{k_l}\Im(a^{k_l}))}{2q}.$$

- $\Rightarrow$

$$\begin{aligned} q &= -\frac{E_1^{k_l}(\Re(a^{k_l}) + \nabla \log E_1^{k_l})\cdot \Im(a^{k_l}) - \nabla \cdot (E_1^{k_l}\Im(a^{k_l}))}{2k_l E_1^{k_l}} \\ &= -\frac{\Re(a^{k_l})\cdot \Im(a^{k_l}) - \nabla \cdot \Im(a^{k_l})}{2k_l}. \end{aligned}$$

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- A^k:

$$A^k = [\nabla \alpha^k_{j+1}]_{j=1}^d ,$$

$$\alpha_j^k = \frac{E_j^k}{E_1^k}$$

and the data $(E_j)_{j=1}^{d+1}$: given by proper set of measurements $\{1, x_1, \ldots, x_d\}$.

- Exact formula:

$$q(x) = \frac{-\Re(a^k) \cdot \Im(a^k) + \nabla \cdot \Im(a^k)}{2k} ;$$

- $a^k = ((A^k)^* A^k)^{-1} [(A^k)^* \nabla \cdot A^k] .$

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- Optimal control approach:
  - Minimization ofthe discrepancy functional:

$$J[q] = \frac{1}{2} \int_{\underline{k}}^{\overline{k}} \int_{\Omega} \left| q|u^k|^2 - E^k \right|^2 dx dk$$

  - Initial guess:

$$q_I = \frac{1}{\overline{k} - \underline{k}} \int_{\underline{k}}^{\overline{k}} \frac{-\Re(a^k) \cdot \Im(a^k) + \nabla \cdot \Im(a^k)}{2k} dk \, .$$

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- Differentiability of the data map and its inverse:
  - Let $0 < \underline{q} < \overline{q}$. Let $L^\infty_+(\Omega)$: open set in $L^\infty(\Omega)$:

$$L^\infty_+(\Omega) = \left\{ p \in L^\infty(\Omega) : \underline{q} < p < \overline{q} \text{ in } \Omega \right\}.$$

  - Fix $k \in (\underline{k}, \overline{k})$. Define the solution and the data map as

$$\begin{array}{rcl} u^k : L^\infty_+(\Omega) & \to & H^1(\Omega) \\ q & \mapsto & u^k[q] \end{array}$$

$$\begin{array}{rcl} F^k : L^\infty_+(\Omega) & \to & L^2(\Omega) \\ q & \mapsto & F^k[q] = q|u^k[q]|^2 , \end{array}$$

  $u^k[q]$: solution of

$$\begin{cases} \Delta u^k + (k^2 + ikq)u^k = 0 \quad \text{in } \Omega , \\ u^k = 1 \quad \text{on } \partial\Omega . \end{cases}$$

  - F^k: well-defined $\leftarrow u^k \in \mathcal{C}^1(\overline{\Omega})$.

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- u^k: Fréchet differentiable in $L^\infty_+(\Omega)$. Derivative at q given by

$$du^k[q](\rho) = v^k(\rho), \quad \forall \rho \in B_q \,,$$

$B_q \subset L^\infty_+(\Omega)$: open neighborhood of 0 (that depends on q) in $L^\infty(\Omega)$ and $v^k(\rho)$: solution of

$$\begin{cases} \Delta v^k + (k^2 + ikq)v^k = -ik\rho u^k[q] \quad \text{in } \Omega \,, \\ v^k = 0 \quad \text{on } \partial\Omega \,. \end{cases}$$

- Differentiability of F^k:
  - (i) F^k: Fréchet differentiable;

$$dF^k[q](\rho) = \rho |u^k[q]|^2 + 2q\Re(u^k[q]\overline{v}^k(\rho)), \quad \forall q \in L^\infty_+(\Omega), \rho \in B_q \,.$$

  - (ii) Dual of dF^k, dF^{k^*}: given by

$$(\rho, dF^{k^*}[q](h)) = \Re \int_\Omega \left(q|u^k[q]|^2 h + iku^k[q]\overline{p}^k(h) \right) \rho dx \ ,$$

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- $p^k(h)$: solution of the dual problem

$$\begin{cases} \Delta p^k + (k^2 + ikq)p^k = 2qu^k[q]h & \text{in } \Omega\,, \\ p^k = 0 & \text{on } \partial\Omega\,. \end{cases}$$

(iii) There exists $c > 0$ s.t. for all $\rho \in L^2(\Omega)$

$$\int_{\underline{k}}^{\overline{k}} \| dF^k[q](\rho) \|_{L^2(\Omega)} dk \geq c \|\rho\|_{L^2(\Omega)}\,.$$

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- Proof: It is sufficient to show that

$$\lim_{\|\rho\|_{L^\infty(\Omega)}\to 0} h(\rho) = 0\,; \quad h(\rho) = \frac{\|u^k[q+\rho] - u^k[q] - v^k(\rho)\|_{L^2(\Omega)}}{\|\rho\|_{L^\infty(\Omega)}}\,.$$

- $u^k[q+\rho] - u^k[q] - v^k(\rho)$ solution to

$$\begin{cases} (\Delta + k^2 + ikq)(u^k[q+\rho] - u^k[q] - v^k(\rho)) = -ik\rho(u^k[q+\rho] - u^k[q]) \text{ in } \Omega\,, \\ u^k[q+\rho] - u^k[q] - v^k(\rho) = 0 \text{ on } \partial\Omega\,. \end{cases}$$

- $\Rightarrow$

$$\|u^k[q+\rho] - u^k[q] - v^k(\rho)\|_{L^2(\Omega)} \leq \frac{\|\rho\|_{L^\infty(\Omega)}\|(u^k[q+\rho] - u^k[q])\|_{L^2(\Omega)}}{\inf q}\,.$$

- $u^k[q+\rho] - u^k[q]$ satisfies

$$\begin{cases} \Delta(u^k[q+\rho] - u^k[q]) + (k^2 + ik(q+\rho))(u^k[q+\rho] - u^k[q]) &= -ik\rho u^k[q] \\ u^k[q+\rho] - u^k[q] &= 0 \end{cases}$$

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- $\Rightarrow$

$$\|u^k[q+\rho] - u^k[q]\|_{L^2(\Omega)} \leq \frac{\|\rho\|_{L^\infty(\Omega)} \|u^k[q]\|_{L^2(\Omega)}}{\inf(q+\rho)}.$$

- Since $dF^k[q] = |u^k[q]|^2(I + \mathrm{compact})$, it is sufficient to prove that

$$\cap_{k \in (\underline{k}, \overline{k})} \mathrm{Ker} dF^k[q] = \{0\}$$

  and then apply the Fredholm alternative.

- Assume there exists $\rho \in L^2(\Omega) \setminus \{0\}$ s.t.

$$dF^k[q](\rho) = 0$$

  for all $k \in (\underline{k}, \overline{k})$. By analyticity, it follows that $dF^0[q](\rho) = 0$. Contradiction: $v^0(\rho) = 0$ and $u^0[q] = 1$.

- Regularity theory $\Rightarrow u^k[q] \in L^\infty(\Omega) \Rightarrow dF^k[q]$ can be extended so that its domain is $L^2(\Omega)$.

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- J: Fréchet differentiable in q. For all $q \in L^\infty_+(\Omega)$,

$$dJ[q](\rho) = \int_{\underline{k}}^{\overline{k}} \int_\Omega \rho \left(|u^k[q]|^2 (q|u^k|^2 - E^k) + \Re ikp^k \overline{u}^k[q] \right) dxdk \, ;$$

  p^k: solution of the dual problem with $h = (q|u^k|^2 - E^k)$.

- Gradient descent method to minimize J:

$$q^{(n+1)} = Tq^{(n)} - \eta dJ[Tq^{(n)}] \, ,$$

  $\eta > 0$: step size,

$$Tf = \max\{\underline{q}, \min\{\overline{q}, f\}\} \, .$$

- Landweber scheme:

$$q^{(n+1)} = Tq^{(n)} - \eta \int_{\underline{\omega}}^{\overline{\omega}} \left[dF^*[Tq^{(n)}] \big(F^k(Tq^{(n)}) - E^k \big) \right] dk \, .$$

- Landweber scheme or equivalently the optimal control approach converges to q_*.