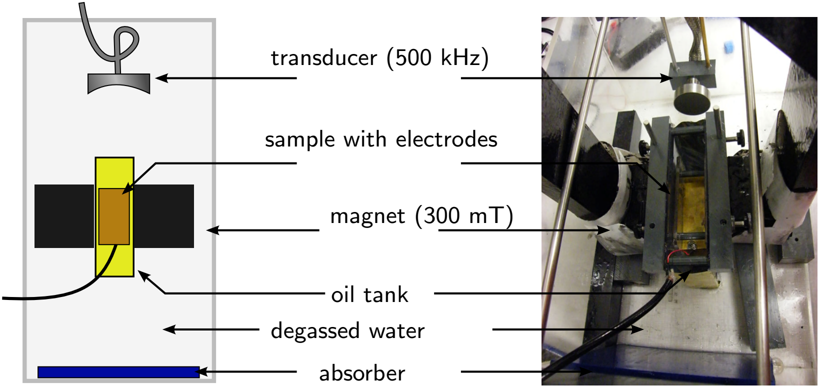
Lecture 7: Ultrasonically-induced Lorentz force imaging

Habib Ammari

Department of Mathematics, ETH Zürich

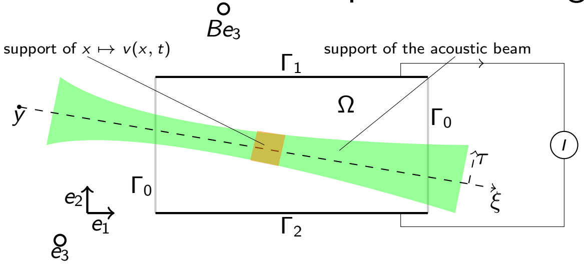
- Mathematical and numerical framework for ultrasonically-induced Lorentz force electrical impedance tomography:
    - Ultrasonic vibration of a tissue in the presence of a static magnetic field $\rightarrow$ electrical current by the Lorentz force.
    - Current: depends nonlinearly on the conductivity distribution.
    - Imaging problem: reconstruct the conductivity distribution from measurements of the induced current.
    - Solve this nonlinear inverse problem:
        - Virtual potential: relate explicitly the current measurements to the conductivity distribution and the velocity of the ultrasonic pulse.
        - Wiener filtering of the measured data: reduce the problem to imaging the conductivity from an internal electric current density.
        - Optimal control approach.
        - Viscosity-type regularization method.

Lorentz force electrical impedance tomography



Example of the imaging device. A transducer is emitting ultrasound in a sample placed in a constant magnetic field. The induced electrical current is collected by two electrodes.

Lorentz force electrical impedance tomography



- Interaction between $v(x,t)\xi$ and Be_3: induces Lorentz' force on the ions in $\Omega \Rightarrow$ separation of charges $\equiv$ source of current and potential: $j_S(x,t) = \frac{B}{e^+}\sigma(x)v(x,t)\tau(\xi)$; $\tau(\xi) = \xi \times e_3$; e^+: elementary charge.
- Voltage potential u:

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = \nabla \cdot j_S \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \; \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0. \end{cases}$$

- Measured intensity: $I(y,\xi) = \displaystyle\int_{\Gamma_2} \sigma \frac{\partial u}{\partial \nu}$.

Lorentz force electrical impedance tomography

- Virtual potential:

$$U := F[\sigma] = \begin{cases} -\nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_1, \\ U = 1 & \text{on } \Gamma_2, \\ \partial_\nu U = 0 & \text{on } \Gamma_0. \end{cases}$$

- Assume that the support of v does not intersect the electrodes Γ_1 and Γ_2.
- Integration by parts $\Rightarrow$

$$-\int_\Omega \sigma \nabla u \cdot \nabla U + \int_{\Gamma_2} \sigma \frac{\partial u}{\partial \nu} = \int_\Omega j_S \cdot \nabla U \, .$$

- $\Rightarrow$

$$I = \int_\Omega j_S \cdot \nabla U \, .$$

Lorentz force electrical impedance tomography

- Link between the measured intensity I and σ:

$$I = \frac{B}{e^+} \int_\Omega v(x,t)\sigma(x)\nabla U(x) dx \cdot \tau \, .$$

- v depends on y, ξ, and t, so does I.
- Define the measurement function:

$$M(y,\xi,z) = \int_\Omega v(x, z/c)\sigma(x)\nabla U(x) dx \cdot \tau(\xi)$$

  for any $y \in \mathbb{R}^3$, $\xi \in S$ and $z > 0$.
- Assume the knowledge of this function in a certain subset of $\mathbb{R}^3 \times S \times \mathbb{R}^+$ denoted by $Y \times \mathfrak{S} \times (0, z_{max})$.

Lorentz force electrical impedance tomography

- Construction of the virtual current: Obtain $\sigma \nabla U$ from $M \leftarrow$ separate v from M.
- Ultrasound pulse:

$$v(x,t) = w(z-ct)\ A(z,|r|);$$

$$z = (x-y)\cdot\xi \text{ and } r = x-y-z\xi\ \in \Upsilon_\xi := \{\zeta \in \mathbb{R}^3\ :\ \zeta\cdot\xi = 0\}.$$

- For any $z \in (0, z_{max})$,

$$\begin{aligned} M(y,\xi,z) &= \int_{\mathbb{R}}\int_{\Upsilon_\xi} w(z-z')(\sigma\nabla U)(y+z'\xi+r)A(z',|r|)drdz'\cdot\tau(\xi) \\ &= \int_{\mathbb{R}} w(z-z')\int_{\Upsilon_\xi}(\sigma\nabla U)(y+z'\xi+r)A(z',|r|)drdz'\cdot\tau(\xi) \\ &= (W\star\Phi_{y,\xi})(z)\cdot\tau(\xi), \end{aligned}$$

$W(z) = w(-z)$; $\star$: convolution product;

$$\Phi_{y,\xi}(z) = \int_{\Upsilon_\xi}\sigma(y+z\xi+r)A(z,|r|)\nabla U(y+z\xi+r)dr.$$

Lorentz force electrical impedance tomography

- Deconvolution:
  - Recover $\Phi_{y,\xi}$ from the measurements $M(y, \xi, \cdot)$ in the presence of noise.
  - Wiener-type filter.
  - Assume that the signal $M(y, \xi, \cdot)$ is perturbed by a random white noise:

$$\widetilde{M}(y, \xi, z) = M(y, \xi, z) + \mu(z),$$

μ: white Gaussian noise with variance ν^2 s.t.

$$\mathbb{E}[\mu(z)\mu(z')] = \nu^2 \delta_0(z - z')$$

and

$$\mathbb{E}[\mathcal{F}(\mu)(k)\overline{\mathcal{F}(\mu)(k')}] = \nu^2 \delta_0(k - k')\,,$$

where

$$\mathcal{F}[\mu](k) = \frac{1}{\sqrt{2\pi}} \int \mu(z) e^{-ikz} dz\,.$$

Lorentz force electrical impedance tomography

- $$\widetilde{M}_{y,\xi}(z) = \left(W \star \Psi_{y,\xi}\right)(z) + \mu(z)\,,$$
  $\Psi_{y,\xi}(z) = \Phi_{y,\xi}(z) \cdot \tau(\xi)$.
- $S(\Psi_{y,\xi}) = \int_{\mathbb{R}} |\mathcal{F}(\Psi_{y,\xi})(k)|^2 dk$: spectral density of $\Psi_{y,\xi}$; $\mathcal{F}$: Fourier transform.
- Wiener deconvolution filter in the frequency domain:
  $$\widehat{L}(k) = \frac{\overline{\mathcal{F}(W)}(k)}{|\mathcal{F}(W)|^2(k) + \frac{\nu^2}{S(\Psi_{y,\xi})}}\,.$$
- Quotient $S(\Psi_{y,\xi})/\nu^2$: signal-to-noise ratio.
- A priori estimate of the signal-to-noise ratio.
- Recover $\Psi_{y,\xi}$ up to a small error by
  $$\widetilde{\Psi}_{y,\xi} = \mathcal{F}^{-1}\left(\mathcal{F}(\widetilde{M})\widehat{L}\right)\,.$$

Lorentz force electrical impedance tomography

- Wiener deconvolution filter: recover $D(x) = (\sigma \nabla U)(x)$ from measured intensities $I(y, \xi)$.
- Recover σ from $D = \sigma \nabla U$.
- Optimal control algorithm.

Lorentz force electrical impedance tomography

- For $a < b$, $L^\infty_{a,b}(\Omega) := \{f \in L^\infty(\Omega) \ : \ a < f < b\}$; Define $\mathcal{F} : L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega) \to H^1(\Omega)$ by

$$\mathcal{F}[\sigma] = U : \begin{cases} \nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega\,, \\ U = 0 & \text{on } \Gamma_1\,, \\ U = 1 & \text{on } \Gamma_2\,, \\ \dfrac{\partial U}{\partial \nu} = 0 & \text{on } \Gamma_0\,. \end{cases}$$

- $d\mathcal{F}$: Fréchet derivative of $\mathcal{F}$. For any $\sigma \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$ and $h \in L^\infty(\Omega)$ s.t. $\sigma + h \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$,

$$d\mathcal{F}[\sigma](h) = v : \begin{cases} \nabla \cdot (\sigma \nabla v) = -\nabla \cdot (h \nabla \mathcal{F}[\sigma]) & \text{in } \Omega\,, \\ v = 0 & \text{on } \Gamma_1 \cup \Gamma_2\,, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_0\,. \end{cases}$$

Lorentz force electrical impedance tomography

- Proof: $w = \mathcal{F}[\sigma + h] - \mathcal{F}[\sigma] - v$ satisfies
$$\nabla \cdot (\sigma \nabla w) = -\nabla \cdot (h \nabla (\mathcal{F}[\sigma + h] - \mathcal{F}[\sigma]))$$
with the same boundary conditions as v.
- Elliptic global control:
$$\|\nabla w\|_{L^2(\Omega)} \leq \frac{1}{\underline{\sigma}} \|h\|_{L^\infty(\Omega)} \|\nabla (\mathcal{F}[\sigma + h] - \mathcal{F}[\sigma])\|_{L^2(\Omega)} .$$
- $\nabla \cdot (\sigma \nabla (\mathcal{F}[\sigma + h] - \mathcal{F}[\sigma])) = -\nabla \cdot (h \nabla \mathcal{F}[\sigma + h]) , \Rightarrow$
$$\|\nabla (\mathcal{F}[\sigma + h] - \mathcal{F}[\sigma])\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\underline{\sigma}}} \|h\|_{L^\infty(\Omega)} \|\nabla \mathcal{F}[\sigma + h]\|_{L^2(\Omega)} .$$
- $\Rightarrow$ There is a positive constant C depending only on Ω s.t.
$$\|\nabla \mathcal{F}[\sigma + h]\|_{L^2(\Omega)} \leq C \sqrt{\frac{\overline{\sigma}}{\underline{\sigma}}} .$$
- $\Rightarrow$
$$\|\nabla w\|_{L^2(\Omega)} \leq C \frac{\sqrt{\overline{\sigma}}}{\underline{\sigma}^2} \|h\|^2_{L^\infty(\Omega)} .$$

Lorentz force electrical impedance tomography

- Minimization of the functional: $J[\sigma] = \frac{1}{2} \int_\Omega |\sigma \nabla \mathcal{F}[\sigma] - D|^2$.
- Gradient of J: For any $\sigma \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$ and $h \in L^\infty(\Omega)$ s.t. $\sigma + h \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$,

$$dJ[\sigma](h) = -\int_\Omega h \Big((\sigma \nabla \mathcal{F}[\sigma] - D - \nabla p) \cdot \nabla \mathcal{F}[\sigma] \Big) ;$$

p: solution to the adjoint problem:

$$\begin{cases} \nabla \cdot (\sigma \nabla p) = \nabla \cdot (\sigma^2 \nabla \mathcal{F}[\sigma] - \sigma D) & \text{in } \Omega \,, \\ p = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \,, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma_0 \,. \end{cases}$$

Lorentz force electrical impedance tomography

- Proof $\mathcal{F}$: Fréchet differentiable $\Rightarrow$ J: Fréchet differentiable. For $\sigma \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$ and $h \in L^\infty(\Omega)$ s.t. $\sigma + h \in L^\infty_{\underline{\sigma},\overline{\sigma}}(\Omega)$,

$$dJ[\sigma](h) = \int_\Omega (\sigma \nabla \mathcal{F}[\sigma] - D) \cdot (h \nabla \mathcal{F}[\sigma] + \sigma \nabla d\mathcal{F}[\sigma](h)) \,.$$

- $\Rightarrow$

$$\int_\Omega \sigma \nabla p \cdot \nabla d\mathcal{F}[\sigma](h) = \int_\Omega (\sigma^2 \nabla \mathcal{F}[\sigma] - \sigma D) \cdot \nabla d\mathcal{F}[\sigma](h) \,.$$

-

$$\int_\Omega \sigma \nabla p \cdot \nabla d\mathcal{F}[\sigma](h) = - \int_\Omega h \nabla \mathcal{F}[\sigma] \cdot \nabla p \,,$$

- $\Rightarrow$

$$dJ[\sigma](h) = \int_\Omega h(\sigma \nabla \mathcal{F}[\sigma] - D - \nabla p) \cdot \nabla \mathcal{F}[\sigma] \,.$$

Lorentz force electrical impedance tomography

- Optimal control algorithm:
    - $\min_\sigma \int_\Omega |\sigma \nabla F[\sigma] - D|^2 +$ regularization term (a prior) :
        - σ: smooth variations out of the discontinuity set $\Rightarrow$ regularized functional:

$$J_\varepsilon[\sigma] = \frac{1}{2} \int_\Omega |\sigma \nabla \mathcal{F}[\sigma] - D|^2 + \varepsilon \left| \sigma \right|_{TV(\Omega)},$$

$\varepsilon > 0$: regularization parameter.

- Nonconvexity (numerically); high sensitivity to noise.

Lorentz force electrical impedance tomography

Direct method

- Assume $U(x) = x_2\ \Gamma_0 \Rightarrow U$: solution of the transport equation:

$$\begin{cases} D^{\perp} \cdot \nabla U = 0 & \text{in } \Omega\,, \\ U = x_2 & \text{on } \partial\Omega\,. \end{cases}$$

- If transport equation: well posed and can be solved $\Rightarrow$ we can reconstruct the virtual potential $U \Rightarrow$

$$\frac{1}{\sigma} = \frac{D \cdot \nabla U}{|D|^2}\,.$$

- First-order equation: really tricky.
- Existence and uniqueness challenging if F: discontinuous.
- Characteristic method: unstable.

Lorentz force electrical impedance tomography

Direct method

- Viscosity-type regularization method:

$$\begin{cases} \nabla \cdot (\varepsilon I + (D^{\perp}(D^{\perp})^T)\nabla U_\varepsilon = 0 & \text{in } \Omega, \\ U_\varepsilon = x_2 & \text{on } \partial\Omega. \end{cases}$$

- Reconstructed image:

$$\frac{1}{\sigma_\varepsilon} := \frac{D \cdot \nabla U_\varepsilon}{|D|^2} \to \frac{1}{\sigma_*} \text{in } L^2$$

as the viscosity parameter $\varepsilon \to 0$; σ_*: true conductivity.

Lorentz force electrical impedance tomography

- $(U_\eta - U)_{\varepsilon>0}$ converges strongly to zero in $H_0^1(\Omega)$.
- Proof:
  - $F := D^\perp$.
  - $(U_\varepsilon - U)_{\varepsilon>0} \rightarrow 0$ weakly.
  - For any $\varepsilon > 0$, $\tilde{U}_\varepsilon := U_\varepsilon - U \in H_0^1(\Omega)$ and satisfies

$$\nabla \cdot \left[\left(\varepsilon I + FF^T \right) \nabla \tilde{U}_\varepsilon \right] = -\varepsilon \Delta U \quad \text{in } \Omega\,.$$

  - Integration by parts $\Rightarrow$

$$\varepsilon \int_\Omega |\nabla \tilde{U}_\varepsilon|^2 + \int_\Omega |F \cdot \nabla \tilde{U}_\varepsilon|^2 = -\varepsilon \int_\Omega \nabla U \cdot \nabla \tilde{U}_\varepsilon.$$

  - $\Rightarrow$

$$\left\| \tilde{U}_\varepsilon \right\|^2_{H_0^1(\Omega)} \leq \int_\Omega |\nabla U \cdot \nabla \tilde{U}_\varepsilon| \leq \|U\|_{H^1(\Omega)} \left\| \tilde{U}_\varepsilon \right\|_{H_0^1(\Omega)} .$$

Lorentz force electrical impedance tomography

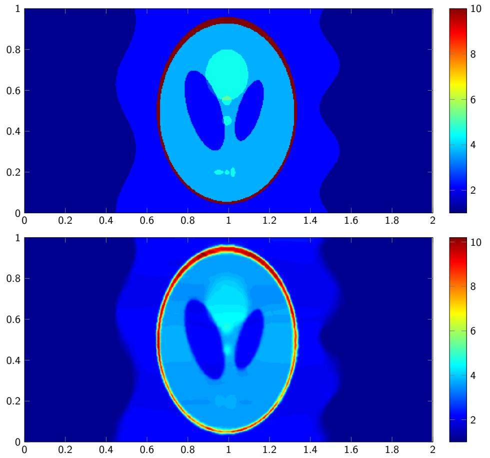
- $\Rightarrow \left\| \tilde{U}_\varepsilon \right\|_{H_0^1(\Omega)} \leq \| U \|_{H^1(\Omega)}$.
- $(\tilde{U}_\varepsilon)_{\varepsilon > 0}$: bounded in $H_0^1(\Omega)$; by Banach-Alaoglu's theorem $\Rightarrow$ extract a subsequence which converges weakly to U^* in $H_0^1(\Omega)$.
- $$\int_\Omega \left(F \cdot \nabla \tilde{U}_\varepsilon \right) (F \cdot \nabla U^*) = -\varepsilon \int_\Omega \nabla U \cdot \nabla U^* - \varepsilon \int_\Omega \nabla \tilde{U}_\varepsilon \cdot \nabla U^* \,.$$
- $\varepsilon \to 0$, $\| F \cdot \nabla U^* \|_{L^2(\Omega)} = 0 \,.$ $\Rightarrow U^*$: solution the transport equation:
$$\begin{cases} F \cdot \nabla U^* = 0 & \text{in } \Omega \,, \\ U^* = 0 & \text{on } \partial\Omega \,. \end{cases}$$
- Uniqueness of a solution $\Rightarrow U^* = 0$ in Ω.
- U^*: independent of the subsequence $\Rightarrow$ convergence holds for $\tilde{U}_\varepsilon$.

Lorentz force electrical impedance tomography

- Strong convergence:
  - $$\int_{\Omega} |\nabla \tilde{U}_{\varepsilon}|^2 \leq - \int_{\Omega} \nabla U \cdot \nabla \tilde{U}_{\varepsilon} \, .$$
  - $\tilde{U}_{\varepsilon} \rightharpoonup 0$ in $H_0^1(\Omega) \Rightarrow \left\| \tilde{U}_{\varepsilon} \right\|_{H_0^1(\Omega)} \to 0$.
- $$\frac{1}{\sigma_{\varepsilon}} = \frac{D \cdot \nabla U_{\varepsilon}}{|D|^2}$$

  strongly converges to $\dfrac{1}{\sigma_*}$ in $L^2(\Omega)$.

Lorentz force electrical impedance tomography



Lorentz force electrical impedance tomography